Pascual Jordan's resolution of the conundrum of the wave-particle duality of light. ⋆

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Abstract

In 1909, Einstein derived a formula for the mean square energy fluctuation in black-body radiation. This formula is the sum of a wave term and a particle term. In a key contribution to the 1926 *Dreimännerarbeit* with Born and Heisenberg, Jordan showed that one recovers both terms in a simple model of quantized waves. So the two terms do not require separate mechanisms but arise from a unified dynamical framework. In this paper, we give a detailed reconstruction of Jordan's derivation of this result and discuss the curious story of its reception. Several authors have argued that various infinities invalidate Jordan's result. We defend it against such criticism. In particular, we note that the fluctuation in a narrow frequency range, which is what Jordan calculated, is perfectly finite. We also note, however, that Jordan's argument is incomplete. In modern terms, Jordan calculated the quantum uncertainty in the energy of a subsystem in an energy eigenstate of the whole system, whereas the thermal fluctuation is the average of this quantity over an ensemble of such states. Still, our overall conclusion is that Jordan's argument is basically sound and that he deserves more credit than he received for having resolved a major conundrum in the development of quantum physics.

Key words: Pascual Jordan, fluctuations, wave-particle duality, *Dreimännerarbeit*, *Umdeutung*, matrix mechanics, quantum field theory

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1 The recovery of Einstein's fluctuation formula in the *Dreimännerarbeit*

In the final section of the famous *Dreimännerarbeit* of Max Born, Werner Heisenberg, and Pascual Jordan (1926), the *Umdeutung* [= reinterpretation] procedure of (Heisenberg, 1925) is applied to a simple system with infinitely many degrees of freedom, a continuous string fixed at both ends. In a lecture in Göttingen in the summer of 1925 (p. 380, note 2) [1] —attended, it seems, by all three authors of the *Dreimännerarbeit*—Paul Ehrenfest (1925) had used this system as a one-dimensional model for a box filled with black-body radiation and had calculated the mean square energy fluctuation in a small segment of it. The string can be replaced by an infinite set of uncoupled harmonic oscillators, one for each mode of the string. The harmonic oscillator is the simplest application of Heisenberg's new quantum-theoretical scheme. The basic idea behind this scheme was to retain the classical equations of motion but to *reinterpret* these equations—hence the term *Umdeutung*—as expressing relations between arrays of numbers, soon to be recognized as matrices (Born and Jordan, 1925), assigned not to individual states but to transitions between them and subject to a non-commutative multiplication law. [2] When this *Umdeutung* procedure is applied to the harmonic oscillators representing the modes of a string and the mean square energy fluctuation in a small segment of the string and in a narrow frequency interval is calculated, one arrives at a surprising result. In addition to the classical wave term, proportional to the square of the mean energy, one finds a term proportional to the mean energy itself. This term is just what one would expect for a system of particles.

For this simple model, one thus recovers both terms of Albert Einstein's well-known formula for the mean square energy fluctuation in a narrow frequency range in a small subvolume of a box with black-body radiation. As Einstein showed in 1909, this formula is required by Planck's law for the spectral distribution of black-body radiation and some general results in statistical mechanics. As Martin J. Klein (1970) characterized the situation: "Einstein concluded that there were two independent causes producing the fluctuations, and that an adequate theory of radiation would have to provide both wave and particle mechanisms" (p. 6). The derivation in the *Dreimännerarbeit* shows, contrary to Einstein's expectation, that both terms in the fluctuation formula can be accounted for within a unified dynamical framework.

The authors presented their unified mechanism in terms of (quantized) waves,

[1] Unless noted otherwise, references are to (Born, Heisenberg, and Jordan, 1926).
[2] See (Duncan and Janssen, 2007) both for an account of what led Heisenberg to this idea and for further references to the extensive historical literature on this subject.

but it can also be described in terms of (quantum) particles. Heisenberg (1930) stated this explicitly a few years later: “The quantum theory, which one can interpret as a particle theory or a wave theory as one sees fit, leads to the complete fluctuation formula” (p. 101; see also, e.g., Jordan, 1936, p. 220). The result thus illustrates the kind of wave-particle duality associated with Niels Bohr’s notion of complementarity, which is different from the kind originally envisioned by Einstein. It does not involve the coexistence of two different mechanisms but the existence of one that can be described in different ways. While illustrating one aspect of complementarity, the fluctuation formula undermines another. A quantum system is supposed to present itself to us either under the guise of waves or under the guise of particles, depending on the experimental context. However, if one were to measure the mean square energy fluctuation in a small subvolume of a box with black-body radiation, as one probably could even though Einstein conceived of it only as a thought experiment, and Einstein’s formula is correct, which is no longer in any serious doubt,[3] one would see the effects of waves and particles simultaneously.[4]

One might have expected that the recovery of Einstein’s fluctuation formula in the *Dreimännerarbeit* would have been hailed right away as the triumphant resolution of a major conundrum in the development of quantum physics; and that it would since have become a staple of historical accounts of the wave-particle duality of light. Both expectations prove to be wrong. As we shall see in sec. 3, Jordan was responsible for this part of the *Dreimännerarbeit* and even his co-authors were skeptical about the result.[5] To give an example from the historical literature, Klein’s (1964) classic paper, “Einstein and the wave-particle duality,” does not even cite the *Dreimännerarbeit.*[6] To the best of our knowledge, the only Einstein biography that touches on the derivation of the fluctuation formula in the *Dreimännerarbeit* is the one by Abraham Pais (1982, p. 405). The canonical twin stories of the light-quantum hypothesis and the wave-particle duality of light end with the Compton effect and Bohr complementarity, respectively. The canonical history fails to mention

[3] *Pace* (Gonzalez and Wergeland, 1973).

[4] We owe this last observation to Jos Uffink (private communication). For discussion of the differences between Heisenberg’s wave-particle equivalence and Bohr’s wave-particle complementarity, see (Camilleri, 2006).

[5] More recently, physicists have recognized the importance of Jordan’s result (see, e.g., Weinberg, 1977, 1995; Wightman, 1996; Cini, 2003).

[6] The same is true for Klein’s (1979, 1980, 1982) contributions to three volumes published in connection with the centenary of Einstein’s birth, even though the first briefly touches on Einstein’s reaction to matrix mechanics (Klein, 1979, p. 149) and the third is specifically on Einstein and fluctuations. In a much earlier paper on Ehrenfest, Klein (1959, p. 50) mentioned the importance of (Ehrenfest, 1925) for this part of the *Dreimännerarbeit*, but added, contrary to what we shall argue, that “a satisfactory discussion of the “mechanism” of the fluctuations” was not given until (Heisenberg, 1931).

that the specific challenge posed by Einstein's fluctuation formula, which suggested wave-particle duality in the first place, was taken up and, we want to argue, convincingly met in the *Dreimännerarbeit*.[7] It also tends to ignore the difference noted above between Einstein's original conception of wave-particle duality and wave-particle duality as it is usually understood in quantum mechanics (see, e.g., Pais, 1982, p. 404).

Having chastised historians of physics in such broad-brush fashion, we hasten to add that ours is certainly not the first contribution to the historical literature that draws attention to the fluctuation calculations in the *Dreimännerarbeit*. For instance, even though Klein (1980) did not mention these calculations in his lecture at the Princeton Einstein centenary symposium, John Stachel, director of the Einstein Papers Project at the time, did bring them up in question time (Woolf, 1980, p. 196). Stachel also drew attention to correspondence between Einstein and Jordan pertaining to these calculations. Unfortunately, most of Einstein's letters to Jordan have not survived.[8] Jagdish Mehra and Helmut Rechenberg (1982–2001, Vol. 3, pp. 149–156) devote a section of their comprehensive history of quantum mechanics to this part of the *Dreimännerarbeit*, although they offer little assistance to a reader having difficulties following the derivation. The relevant section of the *Dreimännerarbeit* also plays a central role in a paper on Einstein's fluctuation formula and wave-particle duality by Alexei Kojevnikov (1990); in a recent paper on Jordan's contributions to quantum mechanics by Jürgen Ehlers (2007); and in a paper on the origin of quantized matter waves by Olivier Darrigol (1986). This last author clearly shares our enthusiasm for these fluctuation calculations, calling them "spectacular" at one point and stressing that they formed the solution

[7] Another episode in the history of Einstein and wave-particle duality that seldom gets attention is the one involving the fraudulent canal ray experiments of Emil Rupp (Van Dongen, 2007a,b).

[8] In response to a query by Stachel, Jordan wrote, whitewashing his own involvement with the Nazis in the process: "Indeed those letters the loss of which I mention in [Jordan, 1969, p. 55] are really destroyed and there is no hope that they could be still discovered anywhere. Perhaps you may be astonished that I did not strive more earnestly to preserve them. But you must understand that only the fact of keeping in my house a series of kind und personal letters of Einstein meant a condition of permanent danger under the circonstances in which I had to live here for "1000 years". Being criticized by Lenard and other political enemies of modern physics as a dangerous follower of Einstein and other antagonists of the Hitler Empire I was forced to await every day the possibility that a police-examination of my papers could be performed and finding there the letters from Einstein might result in my immediate arrest. During the war this danger became still more threatening. Therefore the letters must not only been kept, but they must remain hidden in an appropriate manner, and that was bad for preserving them in cases of air-attack in the night" (Jordan to Stachel, April 14, 1978, typed in imperfect English [Einstein Archive (AE), 75-274]).

to “the most famous puzzle of radiation theory” (ibid., p. 221–222).

Aside from a preamble on Einstein’s fluctuation formula and its early history (sec. 2) and a brief conclusion (sec. 5), our paper is divided into two longer sections, one historical and one technical (secs. 3–4). In the latter, we give a detailed and self-contained reconstruction of the calculation in the *Dreimännerarbeit* of the mean square energy fluctuation in a small segment of a string, Ehrenfest’s simple one-dimensional model for a subvolume of a box filled with black-body radiation. This fills an important gap in the historical literature. Our reconstruction will enable us to defend the derivation against various criticisms and it will enable readers to assess such criticisms and our rejoinders for themselves. In the historical section (sec. 3), which draws both on the work of Darrigol and Kojevnikov and on the results of our own reconstruction of the calculations, we explore the question why the recovery of Einstein’s fluctuation formula in the *Dreimännerarbeit* is not nearly as celebrated as one might have expected it to be.

2 Einstein, fluctuations, and wave-particle duality

In a fifty-page semi-popular history of the light-quantum hypothesis, Jordan (1928, pp. 162–163)[9] distinguished between “Einstein’s first fluctuation formula” and “Einstein’s second fluctuation formula.”[10] The former is derived in the paper introducing light quanta. Einstein (1905) considered black-body radiation in the Wien regime of high frequencies in a box of volume V_0. He imagined a fluctuation as a result of which, for a brief moment, all energy E of the radiation in a narrow frequency range around ν gets concentrated in a subvolume V. Using Boltzmann’s relation between entropy and probability, Einstein showed that the probability of this fluctuation is given by $(V/V_0)^{E/h\nu}$ (where, unlike Einstein in 1905, we used Planck’s constant h). For an ideal gas of N particles in a box of volume V_0, the probability of a fluctuation such that all particles momentarily end up in a subvolume V is given by $(V/V_0)^N$. Comparing the two expressions, Einstein (1905) concluded that “monochromatic radiation [in the Wien regime] behaves thermodynamically as if it consisted of [N] mutually independent energy quanta of magnitude [$h\nu$]” (p. 143).

[9] Jordan sent an offprint of this article to Einstein (Jordan to Einstein, November 23, 1928 [AE 13-476]). From a letter three weeks later (Jordan to Einstein, December 11, 1928 [AE 13-477]), it can be inferred that Einstein replied with a long letter. This letter has not survived (see note 8) and Jordan’s response is of little help in reconstructing its contents.

[10] See (Norton, 2006; Rynasiewicz and Renn, 2006; Uffink, 2006) for recent discussions of Einstein’s early use of fluctuation arguments.

The *Dreimännerarbeit* is concerned with Einstein's second fluctuation formula. This formula does not give the probability of a specific fluctuation but the mean square energy fluctuation of black-body radiation in a narrow frequency range in some subvolume. Statistical mechanics gives the following general formula for the mean square energy fluctuation:

$$\langle \Delta E^2 \rangle = kT^2 \frac{d\langle E \rangle}{dT}, \tag{1}$$

where k is Boltzmann's constant, T is the temperature, and $\langle E \rangle$ is the mean energy. Einstein (1904) derived this formula in the concluding installment of his so-called "statistical trilogy." Unbeknownst to him, it had already been published in 1902 by Josiah W. Gibbs. Einstein (1904, sec. 5), however, was the first to apply it to black-body radiation. A few years later, Einstein (1909a,b) returned to these considerations. As Klein (1964) writes: "instead of trying to derive the distribution law from some more fundamental starting point [as Max Planck had done in 1900], ... [Einstein] assume[d] its correctness and [saw] what conclusions it implied as to the structure of radiation" (p. 9). In the special case of black-body radiation, there is a "frequency specific version" (Rynasiewicz and Renn, 2006, p. 19) of Eq. (1):

$$\langle \Delta E_\nu^2 \rangle = kT^2 \frac{d\langle E_\nu \rangle}{dT} = kT^2 \frac{\partial \rho(\nu, T)}{\partial T} V d\nu, \tag{2}$$

where $\langle E_\nu \rangle = \rho(\nu, T) V d\nu$ is the mean energy of the black-body radiation in the frequency range $(\nu, \nu + d\nu)$ at temperature T in the subvolume V. $\langle \Delta E^2 \rangle$ is the integral of $\langle \Delta E_\nu^2 \rangle$ over all frequencies. By inserting the Rayleigh-Jeans law, the Wien law, and the Planck law (denoted by the subscripts 'RJ', 'W', and 'P') for $\rho(\nu, T)$ in Eq. (2), we find the formula for $\langle \Delta E_\nu^2 \rangle$ predicted by these three laws (c is the velocity of light):

$$\begin{aligned}
\rho_{\mathrm{RJ}} &= \frac{8\pi}{c^3}\nu^2 kT, &\qquad \langle \Delta E_\nu^2 \rangle_{\mathrm{RJ}} &= \frac{c^3}{8\pi\nu^2} \frac{\langle E_\nu \rangle_{\mathrm{RJ}}^2}{V \Delta\nu}; \\
\rho_{\mathrm{W}} &= \frac{8\pi h}{c^3}\nu^3 e^{-h\nu/kT}, &\qquad \langle \Delta E_\nu^2 \rangle_{\mathrm{W}} &= h\nu \langle E_\nu \rangle_{\mathrm{W}}; \\
\rho_{\mathrm{P}} &= \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1}, &\qquad \langle \Delta E_\nu^2 \rangle_{\mathrm{P}} &= \frac{c^3}{8\pi\nu^2} \frac{\langle E_\nu \rangle_{\mathrm{P}}^2}{V \Delta\nu} + h\nu \langle E_\nu \rangle_{\mathrm{P}}.
\end{aligned} \tag{3}$$

For the Rayleigh-Jeans law, $\langle \Delta E_\nu^2 \rangle$ is proportional to the square of the mean energy. To borrow a phrase from John Norton (2006, p. 71), this is the *signature of waves*. For the Wien law, $\langle \Delta E_\nu^2 \rangle$ is proportional to the mean energy itself. This is the *signature of particles*. For the Planck law—originally obtained through interpolation between the Wien law and (what became known as) the Rayleigh-Jeans law—$\langle \Delta E_\nu^2 \rangle$ has both a wave and a particle term.

In a lecture at the 1909 Salzburg *Naturforscherversammlung*, Einstein (1909b) famously prophesized on the basis of this last formula and a similar one for momentum fluctuations "that the next phase of the development of theoretical physics will bring us a theory of light that can be interpreted as a kind of fusion of the wave and emission theories" (pp. 482–483). Contrary to what the term "fusion" [*Verschmelzung*] in this quotation suggests, Einstein believed that his fluctuation formulae called for two separate mechanisms: "the effects of the two causes of fluctuation mentioned [waves and particles] act like fluctuations (errors) arising from *mutually independent causes* (additivity of the terms of which the square of the fluctuation is composed)" (Einstein, 1909a, p. 190, our emphasis).[11] This was still Einstein's view in the early 1920s. As he wrote to Arnold Sommerfeld on October 9, 1921: "I am convinced that some kind of spherical wave is emitted *besides* the directional energetic process" (Einstein, 1987–2006, Vol. 7, p. 486; our emphasis).

Reluctant to abandon the classical theory of electromagnetic radiation and to embrace Einstein's light-quantum hypothesis, several physicists in the 1910s and early 1920s tried either to poke holes in Einstein's derivation of the (energy) fluctuation formula so that they could avoid the formula or to find an alternative derivation of it that avoided light quanta.[12] In the wake of the discovery of the Compton effect and of Satyendra Nath Bose's (1924) new derivation of Planck's black-body radiation law, however, both Einstein's fluctuation formula and his light quanta began to look more and more inescapable. The problem of reconciling the wave and the particle aspects of light thus took on greater urgency. In the paper that provided the simple model of a string used in the *Dreimännerarbeit*, for instance, Ehrenfest (1925) emphasized the paradoxical situation that quantizing the modes of a classical wave according to a method proposed by Peter Debye (1910) gives the correct Planck formula for the spectral distribution of black-body radiation but the wrong formula for the mean square energy fluctuation (Stachel, 1986, p. 379). This problem is also highlighted in the *Dreimännerarbeit* (p. 376). As Einstein characterized the situation in an article on the Compton effect in the *Berliner Tageblatt* of April 20, 1924: "There are ... now two theories of light, both indispensable and—as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists—without any logical connection."[13]

[11] See also (Einstein 1909b, p. 498; 1914, p. 346), the quotation from (Klein, 1970) in the introduction, and the discussion in (Bach, 1989, p. 178).

[12] For historical discussion and further references, see (Bach, 1989) and (Kojevnikov, 1990). For some brief comments, see (Jordan, 1927b, p. 642, note 2), (Born and Jordan, 1930, p. 398, note 1), and (Jordan, 1936, p. 220).

[13] This often-quoted passage can be found, for instance, in (Pais, 1980, 211; 1982, p. 414; 1986, p. 248), (Klein, 1980, p. 182), and (Bach, 1989, p. 182).

One possibility that was seriously considered at the time, especially after the decisive refutation in April 1925 of the theory of Bohr, Kramers, and Slater (1924a), was that light consisted of particles guided by waves (Duncan and Janssen, 2007, sec. 4.2). In this picture, the waves and the particles presumably give separate contributions to the mean square energy fluctuation, just as Einstein had envisioned. A derivation of Einstein's fluctuation formula based on this picture (and Bose statistics) was given by Walther Bothe (1927).[14] At that point, however, the *Dreimännerarbeit*, which is not cited in Bothe's paper, had already shown that the two terms in the fluctuation formula do not require separate mechanisms after all, but can be accounted for within a single unified dynamical framework.[15]

3 Why is the solution to Einstein's riddle of the wave-particle duality of light in the *Dreimännerarbeit* not nearly as famous as the riddle itself?

3.1 One of Jordan's most important contributions to physics

The derivation of the fluctuation formula in the *Dreimännerarbeit*, though presented as part of a collaborative effort, was actually the work of just one of the authors, namely Jordan, "the unsung hero among the creators of quantum mechanics" (Schweber, 1994, p. 5). Today, Jordan is mostly remembered as perhaps the only first-tier theoretical physicist who sympathized strongly and openly with the Nazi ideology.[16] It is hard to say whether this entanglement has been a factor in the neglect of the derivation of the fluctuation formula in the *Dreimännerarbeit*. Our impression is that it only played a minor role. For one thing, it was not until 1930 that Jordan began to voice his Nazi sympathies in print and then only under the pseudonym of Ernst Domaier (Beyler, 2007, p. 71). A much more important factor, it seems, was that Jordan's result immediately met with resistance, even from his co-authors. Right from the start a cloud of suspicion surrounded the result and that cloud never seems

[14] Two earlier papers by Bothe (1923, 1924) and a related paper by Mieczyslaw Wolfke (1921) are cited in the *Dreimännerarbeit* (p. 379, notes 2 and 3).

[15] Independently of full-fledged quantum mechanics and using only Bose's quantum statistics, Reinhold Fürth (1928, p. 312) argued that the fluctuation formula was compatible with waves, particles, or a combination of both. After fleeing Czechoslovakia in 1938, Fürth worked with Born in Edinburgh. In his memoirs, Born (1978, p. 289) praised Fürth's work on fluctuations.

[16] For a detailed recent discussion of this aspect of Jordan's life and career, see (Hoffmann and Walker, 2007).

to have lifted.[17] Our paper can be seen as an attempt to disperse it.

Except for a short period of wavering in 1926, Jordan steadfastly stood by his result and considered it one of his most important contributions to quantum mechanics. He said so on a number of occasions. One such occasion was a conference in honor of Paul A. M. Dirac's 70th birthday. At the conference, Jordan talked about "the expanding earth" (Mehra, 1973, p. 822), a topic that apparently occupied him for 20 years (Kundt, 2007, p. 124). For the proceedings volume, however, he submitted some reminiscences about the early years of quantum mechanics. There he wrote:

> Another piece in the 'Dreimänner Arbeit' gave a result, which I myself have been quite proud of: It was possible to show that the laws of fluctuations in a field of waves, from which Einstein derived the justification of the concept of corpuscular light quanta, can be understood also as consequences of an application of quantum mechanics to the wave field (Jordan, 1973, p. 296).

In the early 1960s, Jordan had likewise told Bartel L. van der Waerden that he "was very proud of this result at the time," adding that he "did not meet with much approval."[18] In a follow-up letter, Jordan wrote:

> What [Born and Jordan 1925] says about radiation is not very profound. But what the *Dreimännerarbeit* says about energy fluctuations in a field of quantized waves is, in my opinion, *almost the most important contribution I ever made to quantum mechanics.*[19]

Jürgen Ehlers (2007), who studied with Jordan, relates: "In the years that I knew him, Jordan rarely talked about his early work. On a few occasions, however, he did tell me that he was especially proud of having derived Einstein's fluctuation formula . . . by quantizing a field" (p. 28).

In late 1925, when the *Dreimännerarbeit* was taking shape, Jordan was probably the only physicist who had done serious work both on the light-quantum hypothesis and on the new matrix mechanics. In his dissertation, supervised by Born and published as (Jordan, 1924), he had criticized the argument for

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[17] For criticism see, e.g., (Smekal, 1926), (Heisenberg, 1931), (Born and Fuchs, 1939a), (Gonzalez and Wergeland, 1973), and (Bach, 1989).

[18] Jordan to Van der Waerden, December 1, 1961. Transcriptions of correspondence between Jordan and Van der Waerden in 1961–1962 can be found in the folder on Jordan in the *Archive for History of Quantum Physics*, cited hereafter as AHQP (Kuhn *et al.*, 1967). Van der Waerden relied heavily on this correspondence in editing his well-known anthology (Van der Waerden, 1968).

[19] Jordan to Van der Waerden, April 10, 1962 (AHQP), our emphasis. In view of the first sentence, it is not surprising that Ch. 4 of (Born and Jordan, 1925), "comments on electrodynamics," was left out of (Van der Waerden, 1968).

ascribing momentum to light quanta in (Einstein, 1917). He had "renounced this heresy" [20] after Einstein (1925) published a brief rejoinder. In another paper, Jordan (1925) showed that he was well versed in the latest statistical arguments concerning light quanta. And in late 1925, Jordan gave Born a manuscript in which he essentially proposed what is now known as Fermi-Dirac statistics. Unfortunately, the manuscript ended up at the bottom of a suitcase that Born took with him to the United States and did not resurface until Born returned from his trip, at which point Jordan had been scooped (Schroer, 2007, p. 49). In addition to his work in quantum statistics, Jordan was one of the founding fathers of matrix mechanics. The *Dreimännerarbeit* is the sequel to (Born and Jordan, 1925), which greatly clarified the theory proposed in Heisenberg's *Umdeutung* paper. Unfortunately for Jordan, few if any physicists at the time were primed for his sophisticated combination of these two contentious lines of research—the statistics of light quanta and matrix mechanics—in his derivation of Einstein's fluctuation formula.

3.2 The reactions of Heisenberg and Born to Jordan's result

Even Jordan's co-authors experienced great difficulty understanding his reasoning and entertained serious doubts about its validity. In the letter cited in note 18, Jordan wrote that "my reduction of light quanta to quantum mechanics was considered misguided [*abwegig*] by Born and Heisenberg for a considerable period of time." In the letter to Van der Waerden cited in note 19, Jordan, after reiterating that these fluctuation considerations were "completely mine" (*ganz von mir*), elaborated on the resistance he encountered from his co-authors

> Later, Heisenberg in fact explicitly questioned whether this application I had made of quantum mechanics to a system of *infinitely many degrees of freedom* was correct. It is true that Born did not second Heisenberg's opinion at the time that it was wrong, but he did not explicitly reject Heisenberg's negative verdict either. [21]

Jordan's recollections fit with statements his co-authors made at various times.

A few weeks before the *Dreimännerarbeit* was submitted, Heisenberg wrote to Wolfgang Pauli: "Jordan claims that the interference calculations come out

[20] P. 13 of the transcript of session 2 of Thomas S. Kuhn's interview with Jordan in June 1963 for the AHQP. For further discussion of (Jordan, 1924, 1925), see session 1, pp. 10–11, 15, and session 2, pp. 16–17 of the interview. For discussion of the section on fluctuations in the *Dreimännerarbeit*, including Jordan's views of the work by Bothe (1923, 1924), see session 3, pp. 8–9.
[21] Jordan to Van der Waerden, April 10, 1962 (AHQP).

right, both the classical [wave] and the Einsteinian [particle] terms ... I am a little unhappy about it, because I do not understand enough statistics to judge whether it makes sense; but I cannot criticize it either, because the problem itself and the calculations look meaningful to me."[22] By 1929, Heisenberg had warmed to Jordan's fluctuation calculations and he included them in his book based on lectures that year at the University of Chicago (Heisenberg, 1930, Ch. V, sec. 7).[23] On the face of it, Heisenberg (1931) lost faith again the following year, when he showed that the mean square energy fluctuation diverges *if we include all possible frequencies*, even in a model that avoids the zero-point energy of the harmonic oscillators of the model used in the *Dreimännerarbeit*. As we shall see in sec. 4, however, Jordan clearly intended to calculate the mean square energy fluctuation *in a narrow frequency range*, even though his notation in various places suggests otherwise. In that case, the result is perfectly finite, regardless of whether we consider a model with or without zero-point energy. It is true that the mean square energy fluctuation as calculated by Jordan diverges when integrated over all frequencies. As Heisenberg showed in his 1931 paper, however, this is essentially an artifact of the idealization that the subvolume for which the energy fluctuations are computed has sharp edges (which necessarily excite arbitrarily high frequency modes). If the edges are smoothed out, the mean square energy fluctuation remains finite even when integrated over all frequencies.[24]

Jordan included his fluctuation argument in (Born and Jordan, 1930, sec. 73, pp. 392–400), a book on matrix mechanics. By the early 1930s, anyone who cared to know must have known that Jordan was responsible for this part of the *Dreimännerarbeit*. This can be inferred, for instance, from Pauli's scathing review of Born and Jordan's book. The reviewer wearily informs his readers that the authors once again trot out the "trains of thought about fluctuation phenomena, which one of the authors (P. Jordan) has already taken occasion to present several times before" (Pauli, 1930). These considerations can indeed be found in (Jordan, 1927b, p. 642; Jordan, 1928, pp. 192–196). They also form the starting point of a review of the current state of quantum electrodynamics at a conference in Charkow the following year (Jordan, 1929, pp. 700–702). We shall quote from these texts below.

That these fluctuation considerations make yet another appearance in (Born and Jordan, 1930) would seem to indicate Born's (belated) approval of Jordan's argument. In the late 1930s, however, in a paper written in exile in

[22] Heisenberg to Pauli, October 23, 1925 (Pauli, 1979, p. 252), quoted (in slightly different translations) and discussed in (Darrigol, 1986, p. 220) and in (Mehra and Rechenberg, 1982–2001, Vol. 3, p. 149).
[23] We already quoted Heisenberg's conclusion in the introduction.
[24] See the discussion following Eq. (53) in sec. 4.2 for further details on Heisenberg's objection and its resolution.

Edinburgh with his assistant Klaus Fuchs,[25] Born sharply criticizes Jordan's argument as well as Heisenberg's 1931 amendment to it. He goes as far as dismissing a central step in the argument in the *Dreimännerarbeit* as "quite incomprehensible reasoning,"[26] offering as his only excuse for signing off on this part of the paper "the enormous stress under which we worked in those exciting first days of quantum mechanics" (Born and Fuchs, 1939a, p. 263).

Born conveniently forgets to mention that he had signed his name to the same argument in the 1930 book. Stress had been a factor during the writing of that book as well. In the fall of 1930, still smarting from Pauli's sarcastic review of the book, Born wrote to Sommerfeld, Pauli's teacher:

> [B]ecause I think very highly of Pauli's accomplishments, I am sorry that our personal relationship is not particularly good. You will probably have realized that on the basis of his criticism of [Born and Jordan, 1930] ... I know full well that the book has major weaknesses, which are due in part to the fact that it was started too early and in part to the fact that I fell ill during the work, a collapse from which unfortunately I still have not fully recovered. But from Pauli's side the nastiness of the attack has other grounds, which are not very pretty.[27]

Born then explains how he had originally asked Pauli to collaborate with him on the development of matrix mechanics and how he had only turned to Jordan after Pauli had turned him down (cf. Born, 1978, pp. 218–219). Ever since, Born continues, Pauli "has had a towering rage against Göttingen and has wasted no opportunity to vent it through mean-spirited comments" (Born to Sommerfeld, October 1, 1930). Born eventually came to agree with the substance of Pauli's criticism of his book with Jordan. In his memoirs, in a chapter written in the early 1960s (Born, 1978, p. 225), Born is very dismissive of the book and concedes that the authors' self-imposed restriction to matrix methods was a "blunder" for which they had rightfully been excoriated by Pauli. In his memoirs, Born's ire is directed not at Pauli but at Jordan, whom he blames for the Göttingen parochialism—or "local patriotism," as he calls it—that led them to use matrix methods only (ibid., p. 230). It is possible that Born had already arrived at this assessment when he attacked Jordan's fluctuation considerations in his paper with Fuchs.

Whatever the case may be, a few months after the publication of their paper, Born and Fuchs (1939b) had to issue a "correction." Pauli's assistant Markus E. Fierz had alerted them to a serious error in their calculations. The resulting

[25] For Born's reaction to Fuchs's later arrest as a Soviet spy, see (Born, 1978, p. 288).

[26] In our reconstruction of Jordan's argument in sec. 4, we shall identify the step that Born and Fuchs found so objectionable (see note 68).

[27] Born to Sommerfeld, October 1, 1930, quoted in Von Meyenn, 2007, pp. 45-47.

two-page "correction" amounts to a wholesale retraction of the original paper. The authors explicitly withdraw their criticism of (Heisenberg, 1931) but do not extend the same courtesy to Jordan. This same pattern returns in Born's memoirs, in another chapter dating from the early 1960s, where Born writes that he and Fuchs "worked on the fluctuations in the black-body radiation but discovered later that Heisenberg had done the same, and better" (Born, 1978, p. 285). We find it hard to suppress the thought that, starting sometime in the 1930s, Born's perception of Jordan and Jordan's work became colored—and who can blame him?—by his former student's manifest Nazi sympathies.

3.3 Jordan's result as an argument for field quantization

As Jordan emphasized, both in the late 1920s and in reflecting on this period later, behind the initial resistance of Born and Heisenberg to his fluctuation calculation was a more general resistance to the notion of quantizing the electromagnetic field. As he told Kuhn: "The idea that from the wave field, i.e., from the electromagnetic field, one had to take another step to quantum mechanics struck many physicists back then as too revolutionary, or too artificial, or too complicated, and they would rather not believe it." [28] The passage from a letter from Jordan to Van der Waerden (see note 21) that we quoted above already hints at this and it is made more explicit as the letter continues:

> I remember that, to the extent that they took notice of these issues at all, other theorists in Göttingen [i.e., besides Born and Heisenberg], [Yakov] Frenkel for instance, considered my opinion, expressed often in conversation, that the electromagnetic field and the Schrödinger field had to be quantized ... as a somewhat fanciful exaggeration or as lunacy. [29] This changed only when Dirac [1927] also quantized both the electromagnetic field and the field of matter-waves. I still remember how Born, who had been the first to receive an offprint of the relevant paper of Dirac, showed it to me and initially looked at it shaking his head. When I then pointed out to him that I had been preaching the same idea all along ever since our *Dreimännerarbeit*, he first acted surprised but then agreed. [30] Heisenberg then also set aside his temporary skepticism, though it was not until considerably later that he himself started to work toward a quantum theory of fields (or "quantum electrodynamics") in the paper he then published with Pauli, [31] which followed up on my three joint papers with Pauli, [Oskar] Klein, and [Eugene]

[28] AHQP interview with Jordan, session 3, p. 8.
[29] "... eine etwas phantastische Uebertreibung oder Verrücktheit."
[30] Jordan's text can be read as saying that Born agreed that Jordan had indeed been championing the same idea, but what he meant, presumably, is that Dirac's paper convinced Born of the merit of the idea.
[31] (Heisenberg and Pauli, 1929, 1930)

> Wigner.[32,33]

These and other publications of the late 1920s make Jordan one of the pioneers of quantum field theory. In his AHQP interview (session 3, p. 9), Jordan told Kuhn the same story he told Van der Waerden. Kuhn asked him in this context who had coined the phrase "second quantization." Jordan told him he had (ibid.). A version of the story he told Van der Waerden and Kuhn in the early 1960s can already be found in a letter to Born of the late 1940s:

> It has always saddened me somehow that the attack on the light-quantum problem already contained in our *Dreimännerarbeit* was rejected by everyone for so long (I vividly remember how Frenkel, despite his very friendly disposition toward me, regarded the quantization of the electromagnetic field as a mild form of insanity[34]) until Dirac took up the idea *from which point onward he was the only one cited in this connection.*[35]

Given the resentment one senses in the italicized clause, there is some irony in how Jordan segues into another version of the same story in the volume in honor of Dirac's 70th birthday:

> *I have been extremely thankful to Dirac in another connection.* My idea that the solution of the vexing problem of Einstein's light quanta might be given by applying quantum mechanics to the Maxwell field itself, aroused the doubt, scepticism, and criticism of several good friends. But one day when I visited Born, he was reading a new publication of Dirac, and he said: 'Look here, what Mr. Dirac does now. He assumes the eigenfunctions of a particle to be non-commutative observables again.' I said: 'Naturally.' And Born said: 'How can you say "naturally"?' I said: 'Yes, that is, as I have asserted repeatedly, the method which leads from the one-particle problem to the many-body problem in the case of Bose statistics' (Jordan, 1973, p. 297; our emphasis).

Since this was a conference honoring Dirac, other speakers can be forgiven for declaring Dirac to be the founding father of quantum field theory. Rudolf Peierls (1973, p. 370) set the tone in his talk on the development of quantum field theory and Julian Schwinger (1973, p. 414) followed suit in his report on quantum electrodynamics the next day. Gregor Wentzel chaired this session and the conference proceedings also contain a reprint of his review of quantum field theory for the Pauli memorial volume, which prominently mentions the *Dreimännerarbeit* and lists the early papers of Jordan and his collaborators in its bibliography (Wentzel, 1960, p. 49 and pp. 74–75). Neither Wentzel

[32] (Jordan and Pauli, 1928; Jordan and Klein, 1927; Jordan and Wigner, 1928).
[33] Jordan to Van der Waerden, April 10, 1962 (AHQP).
[34] "...eine Art leichtes Irresein" [sic].
[35] Jordan to Born, July 3, 1948 (AHQP), our emphasis.

nor Jordan said anything in the discussions following the talks by Peierls and Schwinger. One wonders whether these celebratory distortions of history induced Jordan to submit his reminiscences of the early years of quantum mechanics to the conference proceedings instead of the musings on the expansion of the earth to which he had treated his colleagues at the conference itself.

More recent histories of quantum field theory—(Weinberg, 1977, pp. 19–20) but especially (Darrigol, 1986) and, drawing on Darrigol's work, (Miller, 1994) and (Schweber, 1994)—do full justice to Jordan's contributions.[36] To understand the negative reactions of his co-authors and contemporaries to his derivation of the fluctuation formula it is important to keep in mind that Jordan was virtually alone at first in recognizing the need for the extension of quantum theory to fields.

3.4 The 1926 Smekal interlude

What is suppressed in Jordan's later recollections is that in 1926 he himself started to have second thoughts about second quantization and that (Dirac, 1927) also seems to have been important in dispelling his own doubts.[37] In April 1926, Adolf Smekal published a paper criticizing the fluctuation calculations in the *Dreimännerarbeit*. Smekal argued that, when calculating energy fluctuations in radiation, one should take into account the interaction with matter emitting and absorbing the radiation. Without such interaction, he insisted, the radiation would not reach its equilibrium black-body frequency

[36] See also (Ehlers, 2007) and (Schroer, 2007), specifically on Jordan, as well as (Weinberg, 1995, sec. 1.2, pp. 15–31). It is difficult to gauge both how well-known and how well-understood these calculations have been in the physics community since their publication in 1926. One data point is provided by (Milonni, 1981, 1984). In 1981, this author derived a formula for energy fluctuations in a box of black-body radiation (*not* a subvolume of this box) that has the form of Einstein's 1909 fluctuation formula. He interprets the two terms in his fluctuation formula "in terms of the fundamental processes of spontaneous and stimulated emission, and absorption" and writes that "[t]his interpretation seems obvious in retrospect but has not, to the author's knowledge, been discussed previously" (ibid.). He does not mention the *Dreimännerarbeit*. In a paper on wave-particle duality three years later in a volume in honor of Louis de Broglie's 90th birthday, Milonni (1984, pp. 39–41) does mention the fluctuation calculations in the *Dreimännerarbeit*, though he has clearly missed that these calculations, like Einstein's, pertain to a *subvolume* and seems to be under the impression that they are equivalent to the calculations in (Milonni, 1981). He acknowledges that, when he wrote this 1981 paper, he "was not aware that the Born-Heisenberg-Jordan paper contained a discussion of the fluctuation formula" (Milonni, 1984, p. 62, note 27). Neither were the editors and referees of *American Journal of Physics* it seems.

[37] This interlude is also discussed in (Darrigol, 1986, pp. 222–225).

distribution and would not be detectable so that *a fortiori* fluctuations in its energy would not be observable.

With the first of these two objections, Smekal put his finger on a step that is missing both in H. A. Lorentz's (1916) derivation of the classical formula for the mean square energy fluctuation in black-body radiation, and in Jordan's derivation of its quantum counterpart in the simple model of a string. To derive a formula for *thermal* fluctuations, one needs to consider a thermal ensemble of states. Both Lorentz and Jordan, however, only considered individual states and failed to make the transition to an ensemble of states. A clear indication of the incompleteness of their derivations is that the temperature does not appear anywhere. Smekal is quite right to insist that we consider the system, be it black-body radiation or oscillations in a string, in contact with an external heat bath. This does not mean, however, that the interaction with matter needs to be analyzed in any detail. We can calculate the thermal fluctuations simply assuming that the system has somehow thermalized through interaction with matter. It should also be emphasized that the fluctuations in a small subvolume that Lorentz and Jordan were interested in do not come from the exchange of energy between radiation and matter but from radiation energy entering and leaving the subvolume. Smekal's second objection—that interaction with matter is needed to detect energy fluctuations in radiation—seems to have gained considerable traction with the authors of the *Dreimännerarbeit*, as one would expect given their Machian-positivist leanings.

In response to Smekal's criticism of their paper, the authors retreated to the position that their calculation was certainly valid for sound waves in a solid [38] and that it was still an open question whether it also applied to electromagnetic radiation. This is clear from a paper by Heisenberg (1926, p. 501, note 2) on fluctuation phenomena in crystal lattices and from a letter he simultaneously sent to Born, Jordan, and Smekal. As he told these three correspondents:

> Our treatment [i.e., in the *Dreimännerarbeit*] of fluctuation phenomena is undoubtedly applicable to the crystal lattice ... The question whether this computation of fluctuations can also be applied to a radiation cavity can, as Mr. Smekal emphasizes, not be decided at the moment, as a quantum mechanics of electrodynamical processes has not been found yet. Because of the formal analogy between the two problems (crystal lattice–cavity) I am personally inclined to believe in this applicability, but for now this is just a matter of taste. [39]

A more definite stance would have to await the quantum-mechanical treatment

[38] In his lecture on specific heats at the first Solvay conference in 1911, Einstein (1914, p. 342) had already made it clear that these fluctuation considerations also apply to solids (Bach, 1989, p. 180).
[39] Heisenberg to Born, Jordan, and Smekal, October 29, 1926 (AHQP).

of a full interacting system of radiation and matter. Dirac's paper provided such a treatment.

The retreat triggered by (Smekal, 1926) and the renewed advance after (Dirac, 1927) left some traces in Jordan's writings of this period. Immediately after the discussion of his fluctuation considerations in his semi-popular history of the light-quantum hypothesis, we read:

> For light itself one can look upon the following thesis as the fundamental result of the investigation of Born, Heisenberg, and Jordan, namely that (as demanded by Pauli) a new *field concept* must be developed in which one applies the concepts of quantum mechanics to the oscillating field. But this thesis has in a sense shared the fate of the [fluctuation] considerations by Einstein, the elucidation of which served as its justification: for a long time—even among proponents of quantum mechanics—one either suspended judgement or rejected the thesis. It was accepted only when Dirac showed a year later that Einstein's [1917] laws for emission and absorption for atoms in a radiation field also follow necessarily and exactly from this picture [of quantized fields] (Jordan, 1928, pp. 195–196).

In a footnote appended to the next-to-last sentence, Jordan acknowledges that "this general rejection" of field quantization had caused him to doubt it himself "for a while" and that these doubts had found their way into his two-part overview of recent developments in quantum mechanics (Jordan, 1927a,b). In his presentation of his fluctuation considerations in the second part, Jordan (1927b, pp. 642–643) indeed accepted Heisenberg's criticism (cf. note 21) that it is unclear whether quantum mechanics as it stands applies to systems with an infinite number of degrees of freedom and, again following Heisenberg's lead, retreated to the claim that the analysis certainly holds for a lattice with a finite number of particles. In the *Dreimännerarbeit*, the authors still confidently asserted that the same considerations that apply to a finite crystal lattice "also apply if we go over to the limiting case of a system with infinitely many degrees of freedom and for instance consider the vibrations of an elastic body idealized to a continuum or finally of an electromagnetic cavity" (p. 375). In a note added in proof to his paper the following year, Jordan (1927b, p. 643) announced with obvious relief that Dirac's forthcoming paper completely vindicates the original generalization from a lattice to radiation.

3.5 Jordan's result as evidence for matrix mechanics

The ambivalence of Born and Heisenberg about Jordan's fluctuation considerations is reflected in the use that is made of Jordan's result in the *Dreimännerarbeit*. Rather than hailing it as a seminal breakthrough in understanding the wave-particle duality of light, the authors make it subordinate to the overall aim of promoting matrix mechanics. As they announce at the end of the introduction, the derivation of the fluctuation formula "may well be regarded as significant evidence in favour of the quantum mechanics put forward here" (p. 325). After presenting the result, they comment:

> If one bears in mind that the question considered here is actually somewhat remote from the problems whose investigation led to the growth of quantum mechanics, the result . . . can be regarded as particularly encouraging for the further development of the theory (p. 385).

The one other accomplishment the authors explicitly identify as providing "a strong argument in favour of the theory" is their derivation of the Kramers dispersion formula, "otherwise obtained only on the basis of correspondence considerations" (p. 333). Since the new theory grew directly out of such considerations (Duncan and Janssen, 2007), it is not terribly surprising that it correctly reproduces this formula. The recovery of the Einstein fluctuation formula, which played no role in the construction of the theory, constitutes much more striking evidence.

Moreover, as the authors themselves emphasize, the way in which the theory reproduces the fluctuation formula is a particularly instructive illustration of the basic idea of *Umdeutung*. Before going into the details of the calculations, the authors already express the hope "that the modified kinematics which forms an inherent feature of the theory proposed here would yield the correct value for the interference fluctuations" (p. 377). In the next-to-last paragraph of the paper, they make sure the reader appreciates that this hope has now been fulfilled:

> The reasons leading to the appearance [in the formula for the mean square energy fluctuation] of a term which is not provided by the classical theory are obviously closely connected with the reasons for [the] occurrence of a zero-point energy. The basic difference between the theory proposed here and that used hitherto in both instances lies in the characteristic kinematics and not in a disparity of the mechanical laws. One could indeed perceive one of the most evident examples of the difference between quantum-theoretical kinematics and that existing hitherto on examining [the quantum fluctuation formula], which actually involves no mechanical principles whatsoever (p. 385).

With the exception of the final clause, which is best set aside as a rhetorical flourish, the authors' point is well taken. In the spirit of Heisenberg's groundbreaking paper, "Quantum-theoretical re-interpretation of kinematic and mechanical relations," the fluctuation formula, the Kramers dispersion formula, and other results are obtained not through a change of the dynamical laws (the q's and p's for the oscillators representing the modes of the field satisfy the usual laws of Newtonian mechanics) but through a change of the kinematics (the nature of the q's and p's is changed). In this particular case, this means that, although the wave equation for the string—used here as a proxy for Maxwell's equations—is taken over intact from the classical theory, the displacement of the continuous string from its equilibrium state and the time derivatives of that displacement are no longer given by an infinite set of numbers but rather by an infinite set of infinite-dimensional matrices.

The Hamiltonian for a vibrating string decomposes, both classically and quantum-mechanically, into a sum over infinitely many uncoupled harmonic oscillators. The occurrence of a particle-like term in the quantum formula for the mean square energy fluctuation in a segment of the string is a direct consequence of the zero-point energy of these oscillators. The zero-point energy of the harmonic oscillator had already been derived in the *Umdeutung* paper (Heisenberg, 1925, pp. 271–272; see also Born and Jordan, 1925, pp. 297–300). Stachel (1986, p. 379), in a classic paper on Einstein and quantum physics, correctly identifies the zero-point energy as the key element in Jordan's derivation of the fluctuation formula, but does not mention that the zero-point energy itself is traced to the central new feature of the new theory, the *Umdeutung* of position and momentum as matrices subject to a quantum commutation relation. Without this additional piece of information, it looks as if Jordan obtained his result simply by sleight of hand. Kojevnikov (1990, p. 212) does mention that the zero-point energy is itself a consequence of the new theory, though the point could have done with a little more emphasis. Darrigol (1986, p. 222), in his brief characterization of Jordan's calculation, stresses the role of non-commutativity and does not explicitly mention the zero-point energy at all.[40]

In 1926, Heisenberg, in the letter from which we already quoted in sec. 3.4, made it clear that the fluctuation calculations were important to him only insofar as they provided evidence for matrix mechanics and, by this time, against wave mechanics:

> For the crystal lattice the quantum-mechanical treatment [of fluctuations] undoubtedly means essential progress. This progress is not that one has found the mean square fluctuation; that one already had earlier and is obvi-

[40] Bach (1989, p. 199) acknowledges that "sometimes it is pointed out that the cause of the occurrence of the two terms [in the fluctuation formula] lies in the non-commutativity of the observables of the quantum theory," but claims that this is mistaken since the observables relevant to the fluctuation problem supposedly form an Abelian subalgebra (ibid., pp. 199, 202). This claim is simply false. The relevant observables, the operator for the energy of the whole system and the operator for the energy in part of the system in a narrow frequency range, do not commute (see sec. 4.2, the discussion following eq. (59)).

ous on the basis of general thermodynamical considerations if one introduces quantum jumps. The progress, in fact, is that quantum mechanics allows for the calculation of these fluctuations without explicit consideration of quantum jumps on the basis of relations between q, q' etc. This amounts to a strong argument for the claim that quantum-mechanical matrices are the appropriate means for representing discontinuities such as quantum jumps (something that does not become equally clear in the wave-mechanical way of writing things). The calculation of our *Dreimännerarbeit* thus provided an element of support for the correctness of quantum mechanics.[41]

Now that the calculation had served that purpose, Heisenberg clearly preferred to leave it behind and move on.[42] In fact, he closes his letter reminding his correspondents that "there are so many beautiful things in quantum theory at the moment that it would be a shame if no consensus could be reached on a detail like this" (*eine solche Einzelheit*; ibid.).

3.6 Jordan's result as solving the riddle of the wave-particle duality of light

For Jordan, the value of the fluctuation result as solving the riddle of the wave-particle duality of light was obviously much higher than for Heisenberg. Still, Jordan also had a tendency to make it subservient to a larger cause, albeit field quantization rather than matrix mechanics. We already quoted Jordan (1928, pp. 195–196) saying that the fundamental importance of the result was that it brought out the need to quantize fields. In this same article, however, as in various subsequent publications, Jordan also did full justice to the importance of the result as having resolved the conundrum of the wave-particle duality of light. In his history of the light-quantum hypothesis, he wrote:

> [I]t turned out to be superfluous to explicitly adopt the light-quantum hypothesis: We explicitly stuck to the wave theory of light and only changed the kinematics of cavity waves quantum-mechanically. From this, however, the characteristic light-quantum effects emerged automatically as a consequence ... This is a whole new turn in the light-quantum problem. It is not necessary to include the picture [*Vorstellung*] of light quanta among the assumptions of the theory. One can—and this seems to be the natural way to proceed—start from the wave picture. If one formulates this with the concepts of quantum mechanics, then the characteristic light-quantum effects

[41] Heisenberg to Born, Jordan, and Smekal, October 29, 1926 (AHQP).

[42] In the same letter, Heisenberg claimed that he only reluctantly agreed to the publication of the section on fluctuation phenomena of the *Dreimännerarbeit*: "I wanted to give up on publishing our *Dreimännernote*, because all polemics are abhorrent to me in the bottom of my soul [*weil mir jede Polemik im Grund meiner Seele völlig zuwider ist*] and because I no longer saw any point worth fighting for."

> emerge as necessary consequences from the general laws [*Gesetzmäßigkeiten*] of quantum theory (Jordan, 1928, p. 195).

As Jordan undoubtedly realized, "the characteristic light-quantum effects" referred to in this passage do not include those that involve interaction between the electromagnetic field and matter, such as the photoelectric effect or the Compton effect. His argument only applies to free radiation.[43] As such, however, it does explain why the Einstein fluctuation formula contains both a particle and a wave term and why light quanta are subject to Bose's odd new statistics. That the latter also speaks in favor of Jordan's approach is explicitly mentioned in the *Dreimännerarbeit* (pp. 376–379), in (Jordan, 1928, p. 182), and, in more detail, in Jordan's textbook on quantum mechanics (Jordan, 1936, p. 220).[44] The connection with Bose statistics is not mentioned in (Jordan, 1929), his contribution to the proceedings of the Charkow conference on unified field theory and quantum mechanics. That is probably because the paper is not about quantum statistics but about quantum field theory. As such it provides a prime example of Jordan using his fluctuation result as a means to an end (i.e., the promotion of field quantization), but it also contains a particularly crisp statement of the value of the result as the solution to the riddle of the wave-particle duality of light, complete with an uncharacteristically immodest assessment of its momentous character:

> Einstein drew the conclusion that the wave theory would necessarily have to be replaced or at least supplemented by the corpuscular picture. With our findings [*Feststellungen*], however, the problem has taken a completely different turn. We see that it is not necessary after all to abandon or restrict the wave theory in favor of other models [*Modellvorstellungen*]; instead it just comes down to reformulating [*übertragen*] the wave theory in quantum mechanics. The fluctuation effects, which prove the presence of corpuscular light quanta in the radiation field, then arise automatically as consequences of the wave theory. The old and famous problem how one can understand waves and particles in radiation in a unified manner can thus in principle be considered as taken care of [*erledigt*] (Jordan, 1929, p. 702)

The strong confidence conveyed by that last sentence probably reflects that with (Dirac, 1927) the tide had decisively turned for Jordan's pet project of quantizing fields. Jordan's language in (Born and Jordan, 1930, pp. 398–399) is admittedly more subdued again, but that could be because he feared he would not get a more exuberant statement past his teacher and co-author. As is clear in hindsight (see sec. 3.2), Born was not paying much attention

[43] As mentioned in sec. 3.4, Dirac (1927) first developed the theory for the interaction between the electromagnetic field and matter.

[44] For discussion of the connection to Bose statistics, see (Darrigol, 1986, p. 221). Darrigol quotes from a letter from Jordan to Erwin Schrödinger that can be dated to the summer of 1927, in which Jordan briefly reiterates this point (ibid., p. 224).

and Jordan need not have worried, but Jordan probably did not know that in 1930. In any event, the language of the Charkow proceedings, including the triumphant last sentence, is recycled *verbatim* in (Jordan, 1936, p. 222).

None of these later texts ever drew anywhere near the attention accorded to the *Dreimännerarbeit*. Reading them *seriatim*, moreover, one can appreciate the complaint by Pauli (1930) to the effect that Jordan was starting to sound like a broken record (see sec. 3.2). Jordan did himself a disservice not only by agreeing to present his result merely as a piece of evidence for matrix mechanics in the *Dreimännerarbeit*, but also by trying to make up for that mistake too many times. That Jordan routinely pressed the result into service in his promotion of quantum field theory may also have hurt the recognition of its significance as solving the riddle of the wave-particle duality of light.

3.7 Einstein's reaction to Jordan's result

As the person responsible for the riddle, Einstein should have been especially interested in the solution Jordan claimed to have found. As Jordan told Van der Waerden: "Einstein was really the only physicist from whom I could expect the acknowledgment [*Feststellung*] that with this result a big problem in physics had really been brought to its solution. But, although he reacted very friendly and kindly, Einstein on his part was disinclined to consider matrix mechanics as trustworthy." [45] He told Kuhn the same thing: "One might have imagined that Einstein would have been pleased [with Jordan's result] but Einstein's attitude toward matrix mechanics was that he was having none of it" (AHQP interview with Jordan, session 3, p. 9). [46]

In late October 1925, as the *Dreimännerarbeit* was being completed, Jordan wrote to Einstein enclosing some notes on his fluctuation calculations. [47] Einstein's response has not been preserved (see note 8), but from Jordan's next letter a month and a half later, it can be inferred that Einstein objected to the use of the zero-point energy in the calculation. In his defense, Jordan wrote:

[45] Jordan to Van der Waerden, December 1, 1961 (AHQP). As the reference to Einstein's friendliness suggests, politics did not play a role in Einstein's negative reaction.
[46] In a letter to Ehrenfest of September 30, 1925, Einstein ironically referred to Heisenberg's "large quantum egg" (Fölsing, 1997, p. 566).
[47] Jordan to Einstein, October 29, 1925 (AE 13-473), dated by a reference to Born's departure for the United States the day before. The enclosed notes, it seems, are no longer extant. In the letter, Jordan announces that his paper with Born and Heisenberg will be ready in "8 to 14 days." The *Dreimännerarbeit* was received by *Zeitschrift für Physik* on November 16, 1925.

> My opinion of the zero-point energy of the cavity is roughly that it is really only a formal calculational quantity without direct physical meaning; only the thermal energy referred to $T = 0$ is physically definable. The fluctuations, which formally have been calculated as the mean square fluctuation of thermal energy + zero-point energy, are, of course, *identical with the fluctuations of the thermal energy.*[48]

The answer did not satisfy Einstein. He continued to raise objections to the fluctuation calculations, first in letters to Heisenberg[49] and Ehrenfest,[50] then in a postcard to Jordan. This is the only contribution from Einstein to the correspondence with Jordan that has survived. In it, Einstein objects—to use the terminology that Jordan himself later introduced (see sec. 2)—that matrix mechanics can only reproduce his second fluctuation formula, not the first:

> The thing with the fluctuations is fishy [*faul*]. One can indeed calculate the average magnitude of fluctuations with the zero-point term $\frac{1}{2}h\nu$, but not the probability of a very large fluctuation. For weak (Wien) radiation, the probability, for instance, that all radiation [in a narrow frequency range around ν] is found in a subvolume V of the total volume V_0 is $W = (V/V_0)^{E/h\nu}$. This can evidently not be explained with the zero-point term although the expression is secure [*gesichert*] on thermodynamical grounds.[51]

Presumably, Einstein's problem with the zero-point energy was that, if the energy of each quantum of frequency ν were $\frac{3}{2}h\nu$ rather than $h\nu$, the exponent in the expression for W can no longer be interpreted as the number of light quanta N and black-body radiation in the Wien regime would no longer behave as an ideal gas of N particles. The zero-point energy, however, does not come

[48] Jordan to Einstein, December 15, 1925 (AE 13-474). Jordan inquires whether Einstein had meanwhile received page proofs of the *Dreimännerarbeit*, suggesting that Einstein only had Jordan's notes to go on at this point. Jordan also writes that he is planning to develop "a systematic matrix theory of the electromagnetic field" based on the formalism developed by Born and Norbert Wiener. In view of Jordan's later assessment of the importance of his fluctuation result (see the passages quoted in sec. 3.6), it is interesting note that he wrote to Einstein that such a theory would "still remain far removed from the ideal . . . a more profound light-quantum theory, in which a continuous world and continuous quantities no longer occur at all."

[49] This can be inferred from a letter from Jordan to Einstein that can be dated to February 1926 (AE 13-475), in which Jordan mentions that he has read a letter from Einstein to Heisenberg.

[50] Einstein to Ehrenfest, February 12, 1926, quoted in (Kojevnikov, 1990, p. 212).

[51] Einstein to Jordan, March 6, 1926 (AHQP), quoted, for instance, in (Mehra and Rechenberg, 1982–2001, Vol. 3, p. 156). In a postscript Einstein added somewhat disingenuously: "Other than that, however, I am greatly impressed with matrix theory." The objections raised in this postcard can also be found in the letter to Ehrenfest cited in the preceding note.

into play here, since the energy E in the exponent of Einstein's formula is a thermal average of the excitation energy, the difference between the full energy and the zero-point energy, which Jordan in the quotation above called the "thermal energy."[52]

In his response, Jordan does not return to the issue of the zero-point energy but focuses on the question whether matrix mechanics allows one to calculate the probability of specific fluctuations.[53] He explains that the only way to do this in the theory as it stands is through expansion of such probabilities as power series in the mean square and higher-order fluctuations ($\overline{(E-\overline{E})^2}$, $\overline{(E-\overline{E})^4}$, etc.; cf. Jordan, 1928, p. 194). In October 1927, Jordan revisited both objections in another letter to Einstein.[54] He conceded that the treatment of zero-point energy in the theory remained unsatisfactory and referred Einstein to comments on the issue in a paper he had in the works (Jordan and Pauli, 1928). However, the problem of arbitrary fluctuations, Jordan claimed, had been completely resolved. Referring to work soon to be published as (Jordan, 1927c), received by *Zeitschrift für Physik* on October 11, 1927, he wrote:

> Following the latest papers by Dirac, I have recently studied somewhat more closely the correspondence between a quantized wave field and a system of corpuscular quanta. This closer examination reveals such a perfect internal equivalence between these two systems that one can make the following claim: When, without explicitly appealing to the corpuscular representation, one simply quantizes the oscillations of the radiation field (like I did earlier for the vibrating string), one arrives *in all respects* at the same results as when one proceeds on the basis of the corpuscular representation . . . One then sees immediately that every lawlike regularity understandable on the basis of the corpuscular representation (such as, in particular, your formula $W = (V/V_0)^{E/h\nu}$) is also a necessary consequence of the representation of quantized waves.[55]

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[52] This term is also used in the *Dreimännerarbeit* (p. 377, p. 384). As we shall argue in sec. 4.2, this terminology is somewhat misleading (see the discussion following Eq. (56)).

[53] Jordan to Einstein (AE 13-472), undated but probably written shortly after Einstein's postcard of March 6, 1926.

[54] This letter (AE 13-478) was dated on the basis of a reference to page proofs of (Jordan and Klein, 1927), which was received by *Zeitschrift für Physik* on October 4, 1927.

[55] Jordan to Einstein, October 1927 (AE 13-478). A similar statement can be found in (Jordan, 1927c, pp. 772–774). This paper cites (Dirac, 1926, 1927). In (Born and Jordan, 1930, p. 399), the authors promise that it will be shown in a sequel to the book that a theory of quantized waves correctly reproduces Einstein's first fluctuation formula. Since (Born and Jordan, 1930) was itself the sequel to (Born, 1925), Pauli (1930) began his review by pointing out that "[t]his book is the second volume in a series in which goal and purpose of the nth volume is always made clear

Einstein's response, if there ever was any, has not been preserved. And Jordan never showed in detail how he could recover Einstein's first fluctuation formula. Of course, it is no condemnation of his derivation of the second fluctuation formula that he did not produce a derivation of the first.

Einstein never accepted Jordan's results and maintained to the end of his life that the puzzle of the wave-particle duality of light still had to be solved. As he told Michele Besso a few years before he died: "All these fifty years of conscious brooding have brought me no closer to the answer to the question "What are light quanta?" Nowadays every Tom, Dick, and Harry [*jeder Lump*] thinks he knows it, but he is mistaken." [56] Einstein scholars typically quote such pronouncements approvingly in recounting the story of the light-quantum hypothesis and the wave-particle duality of light. [57] Their message, it seems, is that, as wrong as Einstein turned out to be about other aspects of quantum mechanics, he was *right* about the wave-particle duality of light. In our estimation, he was just stubborn. Quantum electrodynamics provides a perfectly satisfactory solution to Einstein's 1909 riddle of the wave-particle duality of light. Jordan was the first to hit upon that solution. The problem of the infinite zero-point energy, to be sure, is still with us in the guise of the problem of the vacuum energy, but that is a different issue. [58] Recall, moreover, that Jordan avoided the problem of infinite zero-point energy altogether by deriving the mean square energy fluctuation in a finite frequency range.

3.8 Why Jordan's result has not become more famous

In the course of our analysis in secs. 3.1–3.7, we identified several factors that help explain why Jordan's derivation of Einstein's fluctuation formula has not become part of the standard story of wave-particle duality. Before moving on to the actual calculations, we collect these factors here. First, there is the cloud of suspicion that has always surrounded the result. Then there is the tendency, most notably in the case of Heisenberg but also in the case of Jor-

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through the virtual existence of the $(n+1)$th volume." The review helped ensure that, for $n=2$, the $(n+1)$th volume never saw the light of day.

[56] Einstein to Besso, December 12, 1951, quoted, for instance, in (Klein, 1979, p. 133, p. 138).

[57] See, e.g., (Stachel, 1986, pp. 379–380) and (Klein, 1970, pp. 38–39).

[58] For the free-field limit of quantum electrodynamics needed for the fluctuation calculations at issue here, a mathematically precise formulation is obtained by shifting the zero point of energy in the full Hamiltonian to remove the divergent zero-point energy contribution (which is the only divergence exhibited by a free field theory). Since such a shift clearly does not affect the dispersion in the energy, it also does not affect the mean square fluctuation in the energy in a subvolume and in a finite frequency interval.

dan himself, to downplay the value of the result as resolving the conundrum of the wave-particle duality of light and to present it instead as an argument for either matrix mechanics or field quantization. This is reflected in the historical literature where Jordan's result has meanwhile found its proper place in histories of quantum field theory but is hardly ever mentioned in histories of the light-quantum hypothesis or wave-particle duality.

4 Reconstruction of and commentary on Jordan's derivation of Einstein's fluctuation formula

Jordan borrowed a simple model from Ehrenfest (1925, pp. 367–373) to analyze the problem of energy fluctuations in black-body radiation. He considered a string of length l fixed at both ends of constant elasticity and constant mass density. This can be seen as a one-dimensional analogue of an electromagnetic field forced to vanish at the conducting sides of a box. The wave equation for the string—the analogue of the free Maxwell equations for this simple model—is:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{4}$$

where $u(x,t)$ is the displacement of the string at position x and time t and where the velocity of propagation is set equal to unity. The boundary conditions $u(0,t) = u(l,t) = 0$ for all times t express that the string is fixed at both ends. The general solution of this problem can be written as a Fourier series (Ch. 4, Eqs. (41) and (41′)):

$$u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin(\omega_k x), \tag{5}$$

with angular frequencies

$$\omega_k \equiv \frac{k\pi}{l}, \tag{6}$$

and Fourier coefficients (Ch. 4, Eq. (44))

$$q_k(t) = a_k \cos(\omega_k t + \varphi_k). \tag{7}$$

The Hamiltonian for the string is (Ch. 4, Eq. (42) [u^2 should be $\dot{u}^2$]):

$$H = \frac{1}{2}\int_0^l dx \left(\dot{u}^2 + u_x^2\right), \tag{8}$$

where the dot indicates a time derivative and the subscript x a partial derivative with respect to x. The terms $\dot{u}^2$ and u_x^2 are the analogues of the densities of the electric and the magnetic field, respectively, in this simple model of black-body radiation. Inserting Eq. (5) for $u(x,t)$ in Eq. (8), we find (Ch. 4, Eq. (41)):

$$\begin{aligned} H = \frac{1}{2}\int_0^l dx \sum_{j,k=1}^{\infty} & \left(\dot{q}_j(t)\dot{q}_k(t)\sin(\omega_j x)\sin(\omega_k x)\right. \\ & \left. + \ \omega_j\omega_k q_j(t) q_k(t)\cos(\omega_j x)\cos(\omega_k x)\right). \end{aligned} \tag{9}$$

The functions $\{\sin(\omega_k x)\}_k$ in Eq. (5) are orthogonal on the interval $(0,l)$, i.e.,

$$\int_0^l dx \sin(\omega_j x)\sin(\omega_k x) = \frac{l}{2}\delta_{jk}. \tag{10}$$

The same is true for the functions $\{\cos(\omega_k x)\}_k$. It follows that the integral in Eq. (9) will only give contributions for $j = k$ (as can be verified explicitly by substituting l for a in Eq. (18) below). The double sum thus turns into the single sum:

$$H = \sum_{j=1}^{\infty} \frac{l}{4}\left(\dot{q}_j^2(t) + \omega_j^2 q_j^2(t)\right) = \sum_{j=1}^{\infty} H_j. \tag{11}$$

With the help of Eqs. (6)–(7), we find that $H_j = (l/4)a_j^2\omega_j^2 = j^2\pi^2 a_j^2/4l$. It follows that the total energy in the string is finite as long as the amplitudes a_j fall off with j faster than $j^{-3/2}$.

Eq. (11) shows that the vibrating string can be replaced by an infinite number of uncoupled oscillators, one for every mode of the string. This shows that the distribution of the energy over the frequencies of these oscillators is constant in time. Since there is no coupling between the oscillators, there is no mechanism for transferring energy from one mode to another. The spatial distribution of the energy in a given frequency range over the length of the string, however, varies in time. We study the fluctuations of the energy in a narrow frequency range in a small segment of the string. The total energy in that frequency range will be constant but the fraction located in that small segment will fluctuate.

Jordan derived an expression for the mean square energy fluctuation of this energy, first in classical theory, then in matrix mechanics.

4.1 Classical calculation

Changing the upper boundary of the integral in Eq. (9) from l to a ($a \ll l$) and restricting the sums over j to correspond to a narrow angular frequency range $(\omega, \omega + \Delta\omega)$, we find the instantaneous energy in that frequency range in a small segment $(0, a) \subset (0, l)$ of the string. This quantity is simply called E in the paper (cf. Ch. 4, Eq. (43)). We add the subscript (a, ω):

$$E_{(a,\omega)}(t) = \frac{1}{2}\int_0^a dx \sum_{j,k} \big(\dot{q}_j(t)\dot{q}_k(t)\sin(\omega_j x)\sin(\omega_k x) \\ + \ \omega_j\omega_k q_j(t)q_k(t)\cos(\omega_j x)\cos(\omega_k x)\big)\,, \quad (12)$$

where the sums over j and k are restricted to the finite range of integers satisfying $\omega < j(\pi/l) < \omega+\Delta\omega$ and $\omega < k(\pi/l) < \omega+\Delta\omega$. Unless we explicitly say that sums run from 1 to ∞, *all* sums in what follows are restricted to this finite range. This restriction also appears to be in force in many summations in this section of the *Dreimännerarbeit* even though they are all *written* as infinite sums. The sums in Eqs. (43), (45), (46′), (46″), and (47) in the paper (pp. 381–382) should all be over this finite rather than over an infinite range of integers.

There are several clear indications in this section of the paper that the authors are in fact considering a small frequency range. The clearest statement is their description of the situation with black-body radiation which the string is supposed to represent:

> If there is communication between a volume V and a very large volume such that waves which have *frequencies which lie within a small range ν to $\nu + d\nu$* can pass unhindered from one to the other, whereas for all other waves the volumes remain detached, and if E be the energy of the waves with frequency ν in V, then according to Einstein the mean square deviation ... can be calculated (p. 379, our emphasis).

Two pages later, in Eq. (43), the same symbol E is used for what we more explicitly write as $E_{(a,\omega)}$. Immediately below this equation it says in parentheses: "under the explicit assumption that all wavelengths *which come into consideration* are small with respect to a" (p. 381, our emphasis).

The functions $\{\sin(\omega_k x)\}_k$ and the functions $\{\cos(\omega_k x)\}_k$ are not orthogonal on the interval $(0, a)$, so both terms with $j = k$ and terms with $j \neq k$ will contribute to the instantaneous energy $E_{(a,\omega)}(t)$ in Eq. (12). First consider the $(j = k)$ terms. On the assumption that a is large enough for the integrals over $\sin^2(\omega_j x)$ and $\cos^2(\omega_j x)$ to be over many periods corresponding to ω_j, these terms are given by:

$$E^{(j=k)}_{(a,\omega)}(t) \approx \frac{a}{4} \sum_j \left(\dot{q}^2_j(t) + \omega^2_j q^2_j(t) \right) = \frac{a}{l} \sum_j H_j(t). \tag{13}$$

Since we are dealing with a system of uncoupled oscillators, the energy of the individual oscillators is constant. Since all terms $H_j(t)$ are constant, $E^{(j=k)}_{(a,\omega)}(t)$ is constant too and equal to its time average:[59]

$$E^{(j=k)}_{(a,\omega)}(t) = \overline{E^{(j=k)}_{(a,\omega)}(t)}. \tag{14}$$

Since the time averages $\overline{\dot{q}_j(t)\dot{q}_k(t)}$ and $\overline{q_j(t)q_k(t)}$ vanish for $j \neq k$, the $(j \neq k)$ terms in Eq. (12) do not contribute to its time average:

$$\overline{E^{(j\neq k)}_{(a,\omega)}(t)} = 0. \tag{15}$$

The time average of Eq. (12) is thus given by the $(j = k)$ terms:

$$E^{(j=k)}_{(a,\omega)}(t) = \overline{E_{(a,\omega)}(t)}. \tag{16}$$

Combining Eqs. (13) and (16), we see that the time average of the energy in the frequency range $(\omega, \omega + \Delta\omega)$ in the small segment $(0, a)$ of the string is just the fraction (a/l) of the (constant) total amount of energy in this frequency range in the entire string.

From Eq. (16) it follows that the $(j \neq k)$ terms in Eq. (12) give the instantaneous deviation $\Delta E_{(a,\omega)}(t)$ of the energy in this frequency range in the segment $(0, a)$ of the string from its mean (time average) value:

$$\Delta E_{(a,\omega)}(t) \equiv E_{(a,\omega)}(t) - \overline{E_{(a,\omega)}(t)} = E^{(j\neq k)}_{(a,\omega)}(t). \tag{17}$$

We now integrate the $(j \neq k)$ terms in Eq. (12) to find $\Delta E_{(a,\omega)}$. From now on, we suppress the explicit display of the time dependence of $\Delta E_{(a,\omega)}$, q_j and $\dot{q}_j$.

[59] A bar over any quantity denotes the time average of that quantity. The argument that follows, leading to Eqs. (14) and (17) can also be made in terms of averages over the phases φ_k in the Fourier coefficients in Eq. (7).

$$
\begin{aligned}
\Delta E_{(a,\omega)} &= \frac{1}{4}\int\limits_0^a dx \sum_{j\neq k} \left(\dot q_j \dot q_k \left[\cos\left((\omega_j-\omega_k)x\right) - \cos\left((\omega_j+\omega_k)x\right)\right]\right. \\
&\qquad\qquad \left. + \,\omega_j\omega_k q_j q_k \left[\cos\left((\omega_j-\omega_k)x\right) + \cos\left((\omega_j+\omega_k)x\right)\right]\right) \\
&= \frac{1}{4}\sum_{j\neq k}\left(\dot q_j \dot q_k \left[\frac{\sin\left((\omega_j-\omega_k)a\right)}{\omega_j-\omega_k} - \frac{\sin\left((\omega_j+\omega_k)a\right)}{\omega_j+\omega_k}\right]\right. \\
&\qquad\qquad \left. + \,\omega_j\omega_k q_j q_k \left[\frac{\sin\left((\omega_j-\omega_k)a\right)}{\omega_j-\omega_k} + \frac{\sin\left((\omega_j+\omega_k)a\right)}{\omega_j+\omega_k}\right]\right).
\end{aligned}
\tag{18}
$$

Defining the expressions within square brackets as (cf. Ch. 4, Eq. (45′))

$$
\begin{aligned}
K_{jk} &\equiv \frac{\sin\left((\omega_j-\omega_k)a\right)}{\omega_j-\omega_k} - \frac{\sin\left((\omega_j+\omega_k)a\right)}{\omega_j+\omega_k}, \\
K'_{jk} &\equiv \frac{\sin\left((\omega_j-\omega_k)a\right)}{\omega_j-\omega_k} + \frac{\sin\left((\omega_j+\omega_k)a\right)}{\omega_j+\omega_k},
\end{aligned}
\tag{19}
$$

we can write this as (cf. Ch. 4, Eq. (45)):

$$
\Delta E_{(a,\omega)} = \frac{1}{4}\sum_{j\neq k}\left(\dot q_j \dot q_k K_{jk} + \omega_j\omega_k q_j q_k K'_{jk}\right). \tag{20}
$$

Note that both K_{jk} and K'_{jk} are symmetric: $K_{jk} = K_{kj}$ and $K'_{jk} = K'_{kj}$. We now compute the mean square fluctuation of the energy in the segment $(0, a)$ in the frequency range $(\omega, \omega + \Delta\omega)$. Denoting the two parts of the sum in Eq. (20) as $\Delta E_{1_{(a,\omega)}}$ and $\Delta E_{2_{(a,\omega)}}$, respectively, we find (Ch. 4, Eq. (46)):

$$
\overline{\Delta E^2_{(a,\omega)}} = \overline{\Delta E^2_{1_{(a,\omega)}}} + \overline{\Delta E^2_{2_{(a,\omega)}}} + \overline{\Delta E_{1_{(a,\omega)}}\Delta E_{2_{(a,\omega)}}} + \overline{\Delta E_{2_{(a,\omega)}}\Delta E_{1_{(a,\omega)}}}. \tag{21}
$$

Classically, the last two terms are obviously equal to one another. In quantum mechanics we have to be more careful. So it is with malice of forethought that we wrote these last two terms separately. The first two terms are given by (Ch. 4, Eq. (46′))

$$
\begin{aligned}
\overline{\Delta E^2_{1_{(a,\omega)}}} + \overline{\Delta E^2_{2_{(a,\omega)}}} = \frac{1}{16}\sum_{j\neq k}\sum_{j'\neq k'} &\left(\overline{\dot q_j \dot q_k \dot q_{j'} \dot q_{k'}} K_{jk} K_{j'k'}\right. \\
&\left. + \quad \overline{q_j q_k q_{j'} q_{k'}}\,\omega_j\omega_k\omega_{j'}\omega_{k'} K'_{jk} K'_{j'k'}\right);
\end{aligned}
\tag{22}
$$

the last two by (Ch. 4, Eq. (46″))

$$\overline{\Delta E_{1_{(a,\omega)}} \Delta E_{2_{(a,\omega)}}} + \overline{\Delta E_{2_{(a,\omega)}} \Delta E_{1_{(a,\omega)}}} =$$

$$\frac{1}{16} \sum_{j \neq k} \sum_{j' \neq k'} \left(\overline{\dot{q}_j \dot{q}_k q_{j'} q_{k'}} \omega_{j'} \omega_{k'} K_{jk} K'_{j'k'} + \overline{q_j q_k \dot{q}_{j'} \dot{q}_{k'}} \omega_j \omega_k K'_{jk} K_{j'k'} \right). \tag{23}$$

Since $q_k(t) = a_k \cos(\omega_k t + \varphi_k)$ and $\omega_k = k(\pi/l)$ (see Eqs. (6) and (7)), it would seem that the time averages of the products of four q's or four $\dot{q}$'s in Eq. (22) vanish unless $(j = j', k = k')$ or $(j = k', k = j')$. This is not strictly true. Writing $\cos \omega_k t = \frac{1}{2}(e^{i\omega_k t} + e^{-i\omega_k t})$, we see that there will in principle be non-vanishing contributions whenever $\pm \omega_j \pm \omega_k \pm \omega'_j \pm \omega'_k = 0.$[60] In the real physical situation, however, the string will not be *exactly* fixed at $x = 0$ and $x = l$, which means that the ω's will not *exactly* be integral number times (π/l). This removes accidental degeneracies of the form $\pm j \pm k \pm j' \pm k' = 0$ and leaves only the index combinations $(j = j', k = k')$ and $(j = k', k = j')$. These both give the same contribution. Hence (Ch. 4, Eq. (47)):

$$\overline{\Delta E^2_{1_{(a,\omega)}}} + \overline{\Delta E^2_{2_{(a,\omega)}}} = \frac{1}{8} \sum_{j \neq k} \left(\overline{\dot{q}_j^2}\, \overline{\dot{q}_k^2} K_{jk}^2 + \overline{q_j^2}\, \overline{q_k^2} \omega_j^2 \omega_k^2 K'^2_{jk} \right), \tag{24}$$

where we used that, for $j \neq k$, averages of products such as $\overline{\dot{q}_j^2 \dot{q}_k^2}$ are products of the averages $\overline{\dot{q}_j^2}$ and $\overline{\dot{q}_k^2}$. The time averages in Eq. (23), with two q's and two $\dot{q}$'s rather than four q's or four $\dot{q}$'s, vanish even if $(j = j', k = k')$ or $(j = k', k = j')$. These index combinations produce time averages of expressions of the form $\sin(\omega_j t + \varphi_j) \cos(\omega_j t + \varphi_j)$ and these vanish. So, in the classical theory, Eq. (24) gives the total mean square fluctuation.

To evaluate the mean square averages of the q's and $\dot{q}$'s in Eq. (24), we use the virial theorem, which says that the time average of the kinetic energy of any one of the oscillators in Eq. (11) is equal to the time average of its potential energy:

$$\frac{l}{4} \overline{\dot{q}_j^2(t)} = \frac{l}{4} \omega_j^2 \overline{q_j^2(t)} = \frac{1}{2} H_j. \tag{25}$$

It follows that

$$\overline{\Delta E^2_{(a,\omega)}} = \frac{1}{2l^2} \sum_{j \neq k} H_j H_k \left(K_{jk}^2 + K'^2_{jk} \right). \tag{26}$$

[60] This problem does not arise if we consider phase averages instead of time averages. Since the phases φ_j, φ_k, $\varphi_{j'}$, and $\varphi_{k'}$ in the q's and $\dot{q}$'s are statistically independent, the only contributions to Eqs. (22) and (23) with time averages replaced by phase averages come from terms in the quadruple sum over $(j \neq k, j' \neq k')$ with either $(j = j', k = k')$ or $(j = k', k = j')$.

We now assume that the energies H_j of the oscillators of characteristic frequency ω_j vary *smoothly* with j.[61] This assumption, which is not made explicit in the *Dreimännerarbeit*, does not hold for arbitrary distributions of the total energy over the various frequencies, but it does hold for many of them. As long as H_j varies smoothly with j, we can replace the double sum over j and k by a double integral over the continuous variables ω and ω'. Since $\omega_j = (\pi/l)j$ (see Eq. (6)), a sum over j turns into (l/π) times an integral over ω. Next, we introduce the continuous counterparts $K_{\omega\omega'}$ and $K'_{\omega\omega'}$ of K_{jk} and K'_{jk}. They are obtained by changing ω_j and ω_k in Eq. (19) into ω and ω'. When we integrate over the square of $K_{\omega\omega'}$, the contribution coming from the square of second term, which has $\omega+\omega'$ in the denominator, is negligibly small compared to the contribution coming from the square of the first term, which has $\omega - \omega'$ in the denominator (the integral over the product of the first and the second term vanishes). The same is true for integrals over the square of $K'_{\omega\omega'}$.[62] In such integrals $K^2_{\omega\omega'}$ and $K'^2_{\omega\omega'}$ can thus be replaced by the squares of their (identical) first terms. Moreover, if a is very large compared to the wavelengths associated with the frequencies in the narrow range $(\omega, \omega + \Delta\omega)$, we can set (cf. Ch. 4, Eq. (48)):

$$\int d\omega' f(\omega') \frac{\sin^2((\omega - \omega')a)}{(\omega - \omega')^2} = \int d\omega' f(\omega') \pi a \delta(\omega - \omega') = \pi a f(\omega), \tag{27}$$

where $\delta(x)$ is the Dirac delta function and $f(x)$ is an arbitrary function. When we make all these substitutions, Eq. (26) turns into:

$$\overline{\Delta E^2_{(a,\omega)}} = \frac{1}{2l^2} \int d\omega \int d\omega' \left(\frac{l}{\pi}\right)^2 2\pi a \delta(\omega - \omega') H_\omega H_{\omega'} = \frac{a}{\pi} \int d\omega H^2_\omega, \tag{28}$$

where the integrals are over the interval $(\omega, \omega + \Delta\omega)$.[63] Instead of integrating

[61] What we mean by 'smooth' here is that, if the integers j are replaced by real numbers x in the expression for H_j, the result of integrating the function $H(x)$ over some interval of the real numbers is negligibly different from the result of taking the discrete sum of terms H_j over the corresponding range of integers.

[62] It can be shown that neglecting the terms with $(\omega_j + \omega_k)$ in the denominator in these integrals causes a relative error of order $\dfrac{\Delta\omega}{\omega}\dfrac{1}{a\omega}$, a product of two factors much smaller than 1.

[63] The corresponding integrals in Eqs. (47′), (49), and (50) are written as integrals from 0 to ∞, just as the sums in Eqs. (43), (45), (46′), (46″), and (47). After Eq. (49), equivalent to our Eq. (28), and Eq. (50) for what in our notation would be $\overline{E_a}$, the authors write: "In order to obtain [the thermodynamical mean square energy fluctuation and the mean energy] we have merely to *extract those parts referring to* $d\nu = d\omega/2\pi$" (p. 383, our emphasis). This is another clear indication that the authors intended to compute the mean square energy fluctuation in a

we can multiply the integrand by $\Delta\omega = 2\pi\Delta\nu$. If, in addition, we replace functions of ω by functions of ν, we can write Eq. (28) as:

$$\overline{\Delta E^2_{(a,\nu)}} = 2a\Delta\nu H^2_\nu. \tag{29}$$

Finally, we replace H_ν by the average energy in the frequency range $(\nu, \nu+\Delta\nu)$ in the segment $(0, a)$ of the string, using the relation

$$\overline{E_{(a,\nu)}} = N_\nu \left(\frac{a}{l} H_\nu\right), \tag{30}$$

where N_ν is the number of modes in the interval $(\nu, \nu + \Delta\nu)$. Eq. (6) tells us that $\pi N_\nu / l = 2\pi\Delta\nu$, so

$$N_\nu = 2l\Delta\nu. \tag{31}$$

It follows that

$$H_\nu = \frac{\overline{E_{(a,\nu)}}}{2a\Delta\nu}. \tag{32}$$

Inserting this into Eq. (29), we arrive at

$$\overline{\Delta E^2_{(a,\nu)}} = \frac{\overline{E_{(a,\nu)}}^2}{2a\Delta\nu} \tag{33}$$

for the mean square energy fluctuation in the small segment $(0, a)$ of the string in the narrow frequency range $(\nu, \nu+\Delta\nu)$. This is the analogue of the formula for the mean square energy fluctuation in a narrow frequency range in a small part of a larger volume containing black-body radiation. Eq. (33) shows that the mean square fluctuation in the energy is proportional to the mean energy squared.

Eq. (33) holds for any state of the string in which there is a smooth distribution of the total energy over the various modes. However, Eq. (33) is *not* the formula for the *thermal* mean square energy fluctuation, the quantity that should be compared to Einstein's fluctuation formula of 1909. A clear indication of this is that the temperature T does not appear anywhere in its derivation. What we need is not a formula for $\overline{\Delta E^2_{(a,\nu)}}$ in an individual state but a formula for the average $\langle\overline{\Delta E^2_{(a,\nu)}}\rangle$ in a thermal ensemble of states. Without this extra step, the derivation is incomplete. Neither in the classical nor in the quantum-mechanical version of the calculation, did the authors of the *Dreimännerarbeit*

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narrow frequency range.

take this extra step.[64] In their defense, we note that when Lorentz (1916) derived his formula for the mean square fluctuation of the energy in a small subvolume of a box with classical electromagnetic radiation, he derived only the analogue of Eq. (33) for that system and likewise did not calculate the average of this quantity in a thermal ensemble of states.

Unlike the authors of the *Dreimännerarbeit*, we shall calculate the thermal average of the mean square energy fluctuation formulae they derived, both for the classical formula (33) and, at the end of sec. 4.2, for the quantum formula (59). Classically, a state of the string is fully specified by the amplitudes a_k and phases φ_k of the infinite number of modes of the string. The thermal average of any observable $O(a_1, a_2, \ldots \varphi_1, \varphi_2, \ldots)$ of the system, which will be some function of these amplitudes and phases, is given by the average over a canonical ensemble of such states:[65]

$$\langle O(a_1, a_2, \ldots \varphi_1, \varphi_2, \ldots)\rangle = \frac{\displaystyle\sum_{\{a_1,a_2,\ldots\varphi_1,\varphi_2,\ldots\}} O(\ldots)e^{-\beta E_{\{a_1,a_2,\ldots\varphi_1,\varphi_2,\ldots\}}}}{\displaystyle\sum_{\{a_1,a_2,\ldots\varphi_1,\varphi_2,\ldots\}} e^{-\beta E_{\{a_1,a_2,\ldots\varphi_1,\varphi_2,\ldots\}}}}. \tag{34}$$

The underlying physical picture is that we imagine the string to be coupled to an infinite heat bath at temperature T, with Boltzmann factor $\beta \equiv 1/kT$. We compute the ensemble average of the expression for $\overline{\Delta E^2_{(a,\omega)}}$ in Eq. (26). The only part that we need to be careful about is the product $H_j H_k$. So we set O in Eq. (34) equal to:

$$O(a_1, a_2, \ldots \varphi_1, \varphi_2, \ldots) = H_j(a_j, \varphi_j)H_k(a_k, \varphi_k). \tag{35}$$

The energy of the string in a given state is just the sum of the Hamiltonians for all the different modes in that state:

$$E_{\{a_1,a_2,\ldots\varphi_1,\varphi_2,\ldots\}} = \sum_{i=1}^{\infty} H_i(a_i, \varphi_i). \tag{36}$$

It follows that the denominator in Eq. (34) can be rewritten as:

$$\sum_{\{a_1,a_2,\ldots\varphi_1,\varphi_2,\ldots\}} e^{-\beta\sum_{i=1}^{\infty} H_i(a_i,\varphi_i)} = \prod_{i=1}^{\infty}\left(\sum_{\{a_i,\varphi_i\}} e^{-\beta H_i(a_i,\varphi_i)}\right). \tag{37}$$

[64] This omission was also noted by Wightman (1996, p. 150).
[65] Given that the amplitudes and the phases are continuous quantities, the sums in this equation are symbolic representations of integrals with a measure determined by the transformation from $\{q_i, p_i\}$ to $\{a_i, \varphi_i\}$.

For all but the j^{th} and the k^{th} mode, the i^{th} factor in the denominator of Eq. (34) with $O = H_j H_k$ cancels against an identical factor in the numerator. Eq. (34) thus reduces to the product of two factors of the exact same form, one for the j^{th} and one for the k^{th} mode. The j^{th} mode gives:

$$\frac{\displaystyle\sum_{\{a_j,\varphi_j\}} H_j(a_j,\varphi_j) e^{-\beta H_j(a_j,\varphi_j)}}{\displaystyle\sum_{\{a_j,\varphi_j\}} e^{-\beta H_j(a_j,\varphi_j)}}. \tag{38}$$

This is just the ensemble average $\langle H_j \rangle$ of the j^{th} mode. The same is true for the k^{th} mode. The equipartition theorem tells us that the average energy of a one-dimensional simple harmonic oscillator at temperature T is equal to kT. The modes of the string thus satisfy the analogue of the classical Rayleigh-Jeans law for black-body radiation. Using that $\langle H_j H_k \rangle = \langle H_j \rangle \langle H_k \rangle$, we see that the ensemble average of Eq. (26) is given by:

$$\langle \overline{\Delta E^2_{(a,\omega)}} \rangle = \frac{1}{2l^2} \sum_{j \neq k} \langle H_j \rangle \langle H_k \rangle \left(K^2_{jk} + K'^2_{jk} \right). \tag{39}$$

Repeating the steps that got us from Eq. (26) to Eq. (33), we arrive at:

$$\langle \overline{\Delta E^2_{(a,\nu)}} \rangle = \frac{\langle \overline{E_{(a,\nu)}} \rangle^2}{2a\Delta\nu}. \tag{40}$$

This is the classical formula for the *thermal* mean square fluctuation of the energy in a narrow frequency range in a small segment of the string. Note that in this case the assumption we need to make to replace sums by integrals is that $\langle H_i \rangle$ varies smoothly with i, which will certainly be true. In fact, it is a constant: $\langle H_i \rangle = kT$. At first sight, it may be surprising that Eq. (40) for the thermal average of the mean square energy fluctuation has the same form as Eq. (33) for the mean square energy average in an individual state. The reason for this can be gleaned from Eq. (26). The entire contribution to the mean square energy fluctuation comes from off-diagonal (i.e., $j \neq k$) terms involving the product of two distinct, and therefore thermally uncorrelated, modes. This is true for the set of uncoupled oscillators that replaces the string. It is not true for arbitrary systems.

4.2 Quantum-mechanical calculation

In the *Dreimännerarbeit* (pp. 383–384), the classical calculation covered in sec. 4.1 is translated into a quantum-mechanical one with the help of Heisenberg's

Umdeutung procedure. The q's and the $\dot{q}$'s thus become matrices that do not always commute. The zero-point energy of the harmonic oscillator is a direct consequence of this feature. Another consequence is that the terms in Eq. (23), which vanished in the classical case, do contribute to the mean square energy fluctuation in the quantum case. Both this contribution and the zero-point energy of the modes of the string, it turns out, are essential for correctly reproducing the analogue of the particle term in Einstein's fluctuation formula in the simple model used in the *Dreimännerarbeit*.

As we mentioned in the introduction, the key point of Heisenberg's *Umdeutung* paper is that the new quantities representing position and momentum in the new theory still satisfy the classical equations of motion. So the solution for the harmonic oscillator is just the solution of the classical equation of motion for the harmonic oscillator, $\ddot{q}_k(t) = -\omega_k^2 q_k(t)$, but now reinterpreted as an equation for matrices. This solution is given by (Baym, 1969, p. 139, Eq. (530)):[66]

$$q_k(t) = q_k(0)\cos\omega_k t + \frac{2p_k(0)}{l\omega_k}\sin\omega_k t. \tag{41}$$

Differentiating this equation, we find:

$$\dot{q}_k(t) = \frac{2p_k(0)}{l}\cos\omega_k t - \omega_k q_k(0)\sin\omega_k t. \tag{42}$$

In these equations, $q_k(t)$, $\dot{q}_k(t)$, $q_k(0)$, and $p_k(0)$ are all matrices, satisfying the canonical equal-time commutation relations $[q_j(t), q_k(t)] = 0$, $[p_j(t), p_k(t)] = 0$, and $[q_j(t), p_k(t)] = i\hbar\delta_{jk}$.

An energy eigenstate of the system is given by specifying the values of the infinite set $\{n_k\}$ of excitation levels of all the modes of the string. The total energy E of the system in the state $\{n_k\}$ is the expectation value of the Hamilton operator H for the whole system in that state. This is the diagonal matrix

[66] Setting $q_k(0) = a_k\cos\varphi_k$ and $p_k(0) = -(l\omega_k a_k/2)\sin\varphi_k$ in Eq. (41), and interpreting $q_k(t)$, $\dot{q}_k(t)$, $q_k(0)$, and $p_k(0)$ as ordinary numbers, we recover Eq. (7):

$$q_k(t) = a_k\left(\cos\varphi_k\cos\omega_k t - \sin\varphi_k\sin\omega_k t\right) = a_k\cos\left(\omega_k t + \varphi_k\right).$$

In the quantum case, we no longer have the freedom to choose arbitrary phases φ_k that we had in the classical case. Accordingly, we can no longer average over such phases. In the *Dreimännerarbeit* phase averages are simply *defined* as the diagonal part of the quantum-theoretical matrix for the relevant quantity in a basis of energy eigenstates (p. 383).

element $H(\{n_k\},\{n_k\})$ (which in modern notation would be $\langle n_k|H|n_k\rangle$):

$$H(\{n_k\},\{n_k\}) = \sum_{k=1}^{\infty} \left(n_k + \frac{1}{2}\right) \hbar\omega_k. \tag{43}$$

The zero-point energy in Eq. (43) is clearly infinite. However, as long as we continue to restrict ourselves to a narrow frequency range, the contribution to the zero-point energy will be perfectly finite.

To find such quantities as the mean energy and the mean square energy fluctuation in a small part of the string and in a narrow frequency range, we first retrace our steps in the classical calculation given above, keeping in mind that q's, p's, and $\dot{q}$'s are no longer numbers but—in modern terminology—operators. We then evaluate the expectation values of the resulting operators in an energy eigenstate of the full system, specified by the excitation levels $\{n_k\}$. As in Eq. (43), these expectation values are the diagonal matrix elements of the operators in a basis of energy eigenstates. In the *Dreimännerarbeit* the argument is formulated entirely in terms of such matrix elements, but it becomes more transparent if we phrase it in terms of operators and their expectation values. In Ch. 3 of the *Dreimännerarbeit*, on "the connection with the theory of eigenvalues of Hermitian forms," the authors get close to the notion of operators acting on a state space but they do not use it in the more physical sections of the paper. They clearly recognized, however, that the matrix elements they computed are for states specified by excitation levels of the infinite set of oscillators. The final step, which is not in the *Dreimännerarbeit*, is to compute the average of the quantum expectation value of the relevant operator in a canonical ensemble of energy eigenstates.

Most of the intermediate results in the classical calculation can be taken over unchanged with the understanding that we are now dealing with operators rather than numbers. Replacing the q's and $\dot{q}$'s (or, equivalently, the p's) in Eq. (12) for $E_{(a,\omega)}$ by the corresponding operators and renaming the quantity $H_{(a,\omega)}$, we find the Hamilton operator for the small segment $(0,a)$ of the string in the narrow angular frequency range $(\omega,\omega+\Delta\omega)$. This is a perfectly good Hermitian operator, which corresponds, at least in principle, to an observable quantity. We want to emphasize that this is true despite the restriction to a narrow frequency range.

Our first goal is to find the operator $\overline{\Delta H^2_{(a,\omega)}}$ for the mean square fluctuation of $H_{(a,\omega)}$. Eqs. (16) and (17) for the $(j=k)$ terms and the $(j\neq k)$ terms in $E_{(a,\omega)}$, respectively, remain valid for the $(j=k)$ terms and the $(j\neq k)$ terms of $H_{(a,\omega)}$. The $(j=k)$ terms give the operator for the time average of the energy in the segment $(0,a)$ in the frequency range $(\omega,\omega+\Delta\omega)$:

$$H^{(j=k)}_{(a,\omega)} = \overline{H_{(a,\omega)}}. \tag{44}$$

The $(j \neq k)$ terms give the operator for the instantaneous energy fluctuation in this segment and in this frequency range:

$$H^{(j\neq k)}_{(a,\omega)} = \Delta H_{(a,\omega)} = H_{(a,\omega)} - \overline{H_{(a,\omega)}}. \tag{45}$$

This quantity is still given by Eq. (20) as long as the q's and $\dot{q}$'s are read as operators rather than numbers. As before, we split it into two parts, $\Delta H_{(a,\omega)} = \Delta H_{1_{(a,\omega)}} + \Delta H_{2_{(a,\omega)}}$. Thus, as in Eq. (21), the operator $\overline{\Delta H^2_{(a,\omega)}}$ for the mean square fluctuation of the energy in the small segment $(0, a)$ in the frequency range $(\omega, \omega + \Delta\omega)$ is given by four terms. The first two terms are still given by Eq. (26),[67] the last two by Eq. (23) (with E replaced by H). The latter vanished in the classical case but not in the quantum case (p. 384).[68] These terms now give identical contributions for the index combinations $(j = j', k = k')$ and $(j = k', k = j')$ with $j \neq k$ and $j' \neq k'$. The quadruple sum in Eq. (23) reduces to the double sum:

$$\begin{aligned} &\overline{\Delta H_{1_{(a,\omega)}} \Delta H_{2_{(a,\omega)}}} + \overline{\Delta H_{2_{(a,\omega)}} \Delta H_{1_{(a,\omega)}}} \\ &\qquad = \frac{1}{8} \sum_{j \neq k} \left(\overline{\dot{q}_j q_j}\; \overline{\dot{q}_k q_k} + \overline{q_j \dot{q}_j}\; \overline{q_k \dot{q}_k} \right) \omega_j \omega_k K_{jk} K'_{jk}, \end{aligned} \tag{46}$$

where we used that q_j commutes with $\dot{q}_k$ as long as $j \neq k$. The two terms in Eq. (46), it turns out, give identical contributions. We focus on the first. We

[67] As the authors explicitly note, the virial theorem, which was used to get from Eq. (24) to Eq. (26), remains valid in matrix mechanics (pp. 343 and 383).

[68] This is the step that Born and Fuchs (1939a, p. 263) complained involved "quite incomprehensible reasoning" (cf. note 26). They wrote: "The error in the paper of Born, Heisenberg, and Jordan is in the evaluation of the terms $\Delta_1\Delta_2 + \Delta_2\Delta_1$ (see formula (46″) [p. 382; our Eq. (23)]). On [p. 382] it is correctly stated that in the classical calculation the mean value of this quantity over all phases vanishes. This is also true in the quantum mechanical calculation as is apparent from formula (46″). [On the bottom half of p. 384], however, $\overline{\Delta_1\Delta_2 + \Delta_2\Delta_1}$ reappears again with a non-vanishing value [cf. our Eq. (49)] and it is shown that it gives rise to an additional term by means of quite incomprehensible reasoning. It is just this term which transforms the correct formula (2.1) [the mean square energy fluctuation for classical waves; cf. our Eq. (33)] into the thermodynamical formula (1.6) [Einstein's fluctuation formula]. But from the standpoint of wave theory this formula (1.6) is certainly wrong" (ibid.). As we shall see, there is nothing wrong with this step in the argument in the *Dreimännerarbeit*. We suspect that what tripped up Born in 1939 was the distinction between phase averages and time averages in the *Dreimännerarbeit*.

compute the time average $\overline{\dot{q}_j q_j}$. Using Eqs. (41) and (42), we find that

$$\overline{\dot{q}_j q_j} = \overline{\left(\frac{2p_j(0)}{l}\cos\omega_j t - \omega_j q_j(0)\sin\omega_j t\right)\left(q_j(0)\cos\omega_j t + \frac{2p_j(0)}{l\omega_j}\sin\omega_j t\right)},$$

which reduces to

$$\overline{\dot{q}_j q_j} = \frac{1}{l}\left(p_j(0)q_j(0) - q_j(0)p_j(0)\right). \tag{47}$$

Classically, p and q commute, but in quantum theory we have $[q_j(0), p_j(0)] = i\hbar$, so that

$$\overline{\dot{q}_j q_j} = -\frac{i\hbar}{l}. \tag{48}$$

The time average $\overline{q_j \dot{q}_j}$ is likewise given by $i\hbar/l$. Inserting these results into Eq. (46), we find

$$\overline{\Delta H_{1_{(a,\omega)}}\Delta H_{2_{(a,\omega)}}} + \overline{\Delta H_{2_{(a,\omega)}}\Delta H_{1_{(a,\omega)}}} = -\frac{\hbar^2}{4l^2}\sum_{j\neq k}\omega_j\omega_k K_{jk}K'_{jk}. \tag{49}$$

When we add the contributions to $\overline{\Delta H^2_{(a,\omega)}}$ coming from Eq. (49) to those coming from Eq. (26), we find

$$\overline{\Delta H^2_{(a,\omega)}} = \frac{1}{l^2}\sum_{j\neq k}\left(H_j H_k \frac{1}{2}\left(K^2_{jk} + K'^2_{jk}\right) - \frac{\hbar^2}{4}\omega_j\omega_k K_{jk}K'_{jk}\right). \tag{50}$$

Replacing both $\frac{1}{2}\left(K^2_{jk} + K'^2_{jk}\right)$ and $K_{jk}K'_{jk}$ by $\sin^2\left((\omega_j - \omega_k)a\right)/(\omega_j - \omega_k)^2$ (cf. the paragraph before Eq. (27)), we can rewrite Eq. (50) as

$$\overline{\Delta H^2_{(a,\omega)}} = \frac{1}{l^2}\sum_{j\neq k}\left(H_j H_k - \frac{\hbar^2}{4}\omega_j\omega_k\right)\frac{\sin^2\left((\omega_j - \omega_k)a\right)}{(\omega_j - \omega_k)^2}. \tag{51}$$

The next step—and the final step in the *Dreimännerarbeit*—is to evaluate the expectation value of the operator $\overline{\Delta H^2_{(a,\omega)}}$ in the state $\{n_i\}$. This is the diagonal matrix element, $\overline{\Delta H^2_{(a,\nu)}}(\{n_i\},\{n_i\})$.[69] Using that

$$H_j(\{n_i\},\{n_i\}) = \left(n_j + \frac{1}{2}\right)\hbar\omega_j, \tag{52}$$

we find that

$$\begin{aligned}\overline{\Delta E^2_{(a,\omega)}} &\equiv \overline{\Delta H^2_{(a,\omega)}}(\{n_i\},\{n_i\}) \\ &= \frac{1}{l^2}\sum_{j\neq k}\left(\left(n_j+\frac{1}{2}\right)\left(n_k+\frac{1}{2}\right)-\frac{1}{4}\right)\hbar^2\omega_j\omega_k\frac{\sin^2\left((\omega_j-\omega_k)a\right)}{(\omega_j-\omega_k)^2} \\ &= \frac{1}{l^2}\sum_{j\neq k}\left(n_jn_k+\frac{1}{2}\left(n_j+n_k\right)\right)\hbar^2\omega_j\omega_k\frac{\sin^2\left((\omega_j-\omega_k)a\right)}{(\omega_j-\omega_k)^2}. \end{aligned} \tag{53}$$

We thus see that the contribution to the mean square fluctuation coming from the second term in Eq. (50), which comes from the non-commutativity of q and p, cancels the square of the zero-point energy in the contribution coming from the first term.

Eq. (53) also illustrates the problem that Heisenberg (1931) drew attention to a few years later (see sec. 3.2). If we let j and k run from 1 to ∞ instead of restricting them to some finite interval, the double sum in Eq. (53) diverges. The problem comes from the terms with $(n_j + n_k)$; the contribution coming from the terms with n_jn_k will still be perfectly finite, at least after we have made the transition from individual states to a thermal ensemble of states. In that case, the excitation level n_i drops off exponentially with i (see Eq. (60)), so the double sum over the terms with n_jn_k will quickly converge. This is not the case for the terms with just n_j or just n_k. For a fixed value of j, for instance,[70] the double sum over the term with n_j in Eq. (53) will reduce to

[69] We remind the reader that the authors of the *Dreimännerarbeit* do not explicitly distinguish between operators and their expectation values. This is a source of possible confusion at this point. The authors write: "we denote those parts of $\overline{\Delta^2}$ [rendered in bold] which belong to a given frequency ν as $\overline{\Delta^2}$ [not rendered in bold]" (p. 384). Without any further information, one can read this either as a restriction (in our notation) of the operator $\overline{\Delta H^2_a}$ to the operator $\overline{\Delta H^2_{(a,\omega)}}$ or as a restriction of the states $\{n_i\}$ in the matrix element $\overline{\Delta H^2_a}(\{n_i\},\{n_i\})$ to states in which only modes in the frequency interval $\omega < i(\pi/l) < \omega + \Delta\omega$ are present (i.e., $n_i = 0$ for all frequencies $i(\pi/l)$ outside that narrow range). Since the latter reading makes no sense (we are interested in states with excitations over the whole frequency spectrum), we assume that the former reading is what the authors had in mind. We are grateful to Jürgen Ehlers for alerting us to this ambiguity.

[70] For fixed values of k, we run into the same problem.

the single sum:

$$\frac{n_j \hbar^2 \omega_j}{2l^2} \sum_{k=1\ (k\neq j)}^{\infty} \omega_k \frac{\sin^2\left((\omega_j-\omega_k)a\right)}{(\omega_j-\omega_k)^2}. \tag{54}$$

This sum is logarithmically divergent. Following Heisenberg's suggestion in 1931, we can remedy this divergence if we replace the sharp edge of the segment of the string at a by a smooth edge. Integration over the segment $(0, a)$ of the string is equivalent to integration over the whole string if we multiply the integrand by the theta-function $\vartheta(a-x)$ (defined as: $\vartheta(\xi) = 0$ for $\xi < 0$ and $\vartheta(\xi) = 1$ for $\xi \geq 0$). The Fourier coefficients for this theta-function do not fall off fast enough if j or k go to infinity. This is why the factors K_{jk} and K'_{jk} in Eq. (19) do not fall off fast enough either if j or k go to infinity. If we replace the theta-function by a smooth, infinitely differentiable function, the problem disappears, since in that case the Fourier transform will fall off faster than any power of the transform variables j or k. We emphasize that as long as the sums in Eq. (53) are restricted to a finite frequency interval, the result is finite without any such remedy.

As in the classical calculation (cf. Eqs. (26)–(29)), we make the transition from sums to integrals. We can do this as long as the excitation levels n_j vary smoothly with j. We can then replace the double sum over j and k by $(l/\pi)^2$ times a double integral over ω and ω' and n_j and n_k by n_ω and $n_{\omega'}$. We can also replace $\sin^2\left((\omega-\omega')a\right)/(\omega-\omega')^2$ by $\pi a \delta(\omega-\omega')$ (see Eq. (27)). Eq. (53) then turns into:

$$\begin{aligned} \overline{\Delta E^2_{(a,\omega)}} &= \frac{a}{\pi}\int d\omega \int d\omega'\ \delta(\omega-\omega')\left(n_\omega n_{\omega'} + \frac{1}{2}\left(n_\omega + n_{\omega'}\right)\right)\hbar^2\omega\omega' \\ &= \frac{a}{\pi}\int d\omega \left(n_\omega^2 + n_\omega\right)\hbar^2\omega^2. \end{aligned} \tag{55}$$

Replacing integration over the interval $(\omega, \omega+\Delta\omega)$ by multiplication by $\Delta\omega = 2\pi\Delta\nu$ and writing all quantities as functions of ν rather than ω, we find:

$$\overline{\Delta E^2_{(a,\nu)}} \equiv \overline{\Delta H^2_{(a,\nu)}}(\{n_\nu\},\{n_\nu\}) = 2a\Delta\nu\left((n_\nu h\nu)^2 + (n_\nu h\nu)h\nu\right). \tag{56}$$

We now introduce the excitation energy, the difference between the total energy and the zero-point energy. Jordan and his co-authors call this the "thermal energy" (p. 377, p. 384). Although the intuition behind it is clear (cf. note 73), this terminology is misleading. The term 'thermal energy' suggests that the authors consider a thermal ensemble of energy eigenstates, what we would call a mixed state, while in fact they are dealing with individual energy eigenstates, i.e., pure states. We therefore prefer the term 'excitation energy'.

The excitation energy E_ν in the narrow frequency range $(\nu, \nu + \Delta\nu)$ in the entire string in the state $\{n_\nu\}$ is:

$$E_\nu = N_\nu(n_\nu h\nu) = 2l\Delta\nu(n_\nu h\nu), \tag{57}$$

where we used that $N_\nu = 2l\Delta\nu$ is the number of modes between ν and $\nu + \Delta\nu$ (see Eq. (31)). On average there will be a fraction a/l of this energy in the small segment $(0, a)$ of the string (p. 384, equation following Eq. (54)):[71]

$$\overline{E_{(a,\nu)}} = \frac{a}{l} E_\nu = 2a\Delta\nu(n_\nu h\nu). \tag{58}$$

Substituting $\overline{E_{(a,\nu)}}/2a\Delta\nu$ for $n_\nu h\nu$ in Eq. (56), we arrive at the final result of this section of the *Dreimännerarbeit* (Ch. 4, Eq. (55)):

$$\overline{\Delta E^2_{(a,\nu)}} = \frac{\overline{E_{(a,\nu)}}^2}{2a\Delta\nu} + \overline{E_{(a,\nu)}} h\nu. \tag{59}$$

Like Eq. (33) in the classical case, Eq. (59) holds for any state with a smooth distribution of energy over frequency. Unlike the classical formula, however, Eq. (59) has exactly the same form as Einstein's fluctuation formula of 1909 (see the third line of Eq. (3)). The first term has the form of the classical wave term (cf. Eq. (33)); the second term has the form of a particle term.

As in the classical case, however, we are not done yet. Eq. (59), like Eq. (33), is for individual states, whereas what we need is a formula for a thermal ensemble of states. In quantum mechanics, this transition from an individual state (a pure state) to an ensemble of states (a mixed state) is a little trickier than in classical theory. Before we show how this is done, we want to make some comments about the interpretation of Eq. (59) to make it clear that this formula does not give the *thermal* mean square energy fluctuation. In modern terms, the formula is for the mean square *quantum uncertainty* or *quantum dispersion* in the energy in a narrow frequency range in a small segment of the string when the whole string is in an energy eigenstate $\{n_\nu\}$. The operators H and $H_{(a,\nu)}$ do not commute. The system is in an eigenstate of the full Hamiltonian H but in a superposition of eigenstates of the Hamiltonian $H_{(a,\nu)}$ of the subsystem. Eq. (59) is a measure for the spread in the eigenvalues of the eigenstates of $H_{(a,\nu)}$ that make up this superposition rather than a measure of the spread in the value of the energy in the subsystem in an thermal ensemble of eigenstates of the system as a whole. It is that latter spread that gives the thermal mean square energy fluctuation.

[71] The time average $\overline{E_{(a,\nu)}}$ of the excitation energy in the narrow frequency range $(\nu, \nu + \Delta\nu)$ in the small segment $(0, a)$ of the string in the state $\{n_\nu\}$ is the expectation value of the operator $\overline{H_{(a,\nu)} - \frac{1}{2}h\nu}$ in that state.

Proceeding as we did in the classical case (see Eqs. (34)–(40)), we make the transition from the formula for the mean square quantum uncertainty of the energy of the subsystem in an energy eigenstate of the whole system to the formula for the mean square fluctuation of this quantity in an ensemble of such states. As in the classical case, it turns out that these two formulae have the same form. The reason for this is once again that the entire contribution to the mean square energy fluctuation in Eq. (59) comes from off-diagonal (i.e., $j \neq k$) terms involving the product of two distinct, and therefore thermally uncorrelated, modes, as can clearly be seen, for instance, in Eq. (51). The thermal average of the product $H_j H_k$ is the product of the thermal averages of H_j and H_k. This is a special feature of the system of uncoupled harmonic oscillators that we are considering and will not hold in general. In this special case, it turns out, we get from the formula for a single state to the formula for a thermal ensemble of states simply by replacing the excitation levels n_i in Eq. (53) by the thermal excitation levels given by the Planck function.[72]

$$\hat{n}_j \equiv \frac{1}{e^{kT/h\nu_j} - 1}, \tag{60}$$

and repeat the steps that took us from Eq. (53) to Eq. (59).

We now show this in detail, taking Eq. (51) as our starting point. We imagine the string to be coupled to an infinite external heat bath at temperature T. The value of some observable in thermal equilibrium is given by the canonical-ensemble expectation value of the diagonal matrix elements of the corresponding operator O in eigenstates $\{n_i\}$ of the Hamiltonian for the system as a whole:

$$\langle O(\{n_i\},\{n_i\})\rangle = \frac{\sum\limits_{\{n_i\}} O(\{n_i\},\{n_i\}) e^{-\beta E_{\{n_i\}}}}{\sum\limits_{\{n_i\}} e^{-\beta E_{\{n_i\}}}}, \tag{61}$$

where $E_{\{n_i\}} = \sum_{n_i} \left(n_i + \frac{1}{2}\right) \hbar\omega_i$ (see Eq. (43)).[73] We calculate the thermal average of the diagonal matrix elements of the operator $\overline{\Delta H^2_{(a,\omega)}}$ in the state

[72] The criticism at this point in the *Dreimännerarbeit* (p. 379) of the statistics that Debye (1910) used to recover the Planck function (see also Jordan to Einstein, October 29, 1925 [AE 13-473]) is retracted in (Jordan, 1928, p. 182, note).

[73] It does not matter for the ensemble average whether or not we include the zero-point energy in $E_{\{n_i\}}$, since the contributions from the zero-point energy to numerator and denominator are the same and cancel. This clearly is what Jordan was getting at when he introduced the term 'thermal energy' for what we proposed to call the excitation energy (see the passage from his letter to Einstein of December 15, 1925, quoted in sec. 3.7).

$\{n_i\}$: $\langle\overline{\Delta H^2_{(a,\omega)}}(\{n_i\},\{n_i\})\rangle$. The only non-trivial part of this calculation is to determine the thermal average of the matrix elements $H_jH_k(\{n_i\},\{n_i\})$ (with $j \neq k$). These matrix elements are given by (cf. Eq. (53)):

$$H_jH_k(\{n_i\},\{n_i\}) = \left(n_j + \frac{1}{2}\right)\hbar\omega_j\left(n_k + \frac{1}{2}\right)\hbar\omega_k. \tag{62}$$

For the thermal average of this expression, we find, using Eq. (61):

$$\langle H_jH_k(\{n_i\},\{n_i\})\rangle = \frac{\sum\limits_{\{n_i\}}\left(n_j + \frac{1}{2}\right)\hbar\omega_j\left(n_k + \frac{1}{2}\right)\hbar\omega_k e^{-\beta E_{\{n_i\}}}}{\sum\limits_{\{n_i\}} e^{-\beta E_{\{n_i\}}}}. \tag{63}$$

The sum over all possible states $\{n_i\}$ in the denominator can be written as a product of sums over all possible values of the excitation level n_i for all modes i:

$$\sum_{\{n_i\}} e^{-\beta E_{\{n_i\}}} = \prod_{i=1}^{\infty}\left(\sum_{n_i=1}^{\infty} e^{-\beta\left(n_i + \frac{1}{2}\right)\hbar\omega_i}\right). \tag{64}$$

For all but the j^{th} and the k^{th} mode the i^{th} factor in the denominator cancels against an identical factor in the numerator. Eq. (63) thus reduces to a product of two factors of the same form, one for the j^{th} mode and one for the k^{th} mode. Consider the former:

$$\frac{\sum\limits_{n_j}\left(n_j + \frac{1}{2}\right)\hbar\omega_j e^{-\beta(n_j+\frac{1}{2})\hbar\omega_j}}{\sum\limits_{n_j} e^{-\beta(n_j+\frac{1}{2})\hbar\omega_j}} = \frac{1}{2}\hbar\omega_j + \frac{\sum\limits_{n_j} n_j\hbar\omega_j e^{-\beta n_j\hbar\omega_j}}{\sum\limits_{n_j} e^{-\beta n_j\hbar\omega_j}}. \tag{65}$$

The expression in the denominator in the second term on the right-hand side is a geometric series, which we shall call Z:

$$Z \equiv \sum_{n_j} e^{-\beta n_j\hbar\omega_j} = \frac{1}{1 - e^{-\beta\hbar\omega_j}}. \tag{66}$$

The fraction of the two sums in Eq. (65) is just minus the derivative of $\ln Z$ with respect to β. Eq. (66) allows us to write this as:

$$-\frac{d}{d\beta}\ln Z = -\frac{1}{Z}\frac{dZ}{d\beta} = -\left(1 - e^{-\beta\hbar\omega_j}\right)\frac{-\hbar\omega_j e^{-\beta\hbar\omega_j}}{\left(1 - e^{-\beta\hbar\omega_j}\right)^2} = \frac{\hbar\omega_j}{e^{\beta\hbar\omega_j} - 1}, \tag{67}$$

which is equal to $\hat{n}_j\hbar\omega_j$, where we used Eq. (60) for the thermal excitation levels. The right-hand side of Eq. (65) thus becomes $\left(\hat{n}_j + \frac{1}{2}\right)\hbar\omega_j$. Using this result for the j^{th} mode and a similar result for k^{th} mode, we can write Eq. (63) as

$$
\begin{aligned}
\langle H_j H_k(\{n_i\},\{n_i\})\rangle &= \langle H_j(\{n_i\},\{n_i\})\rangle\langle H_k(\{n_i\},\{n_i\})\rangle \\
&= \left(\hat{n}_j + \frac{1}{2}\right)\hbar\omega_j\left(\hat{n}_k + \frac{1}{2}\right)\hbar\omega_k.
\end{aligned}
\tag{68}
$$

Using this result, we calculate the thermal average of the diagonal matrix elements in the state $\{n_i\}$ of the operator in Eq. (51) for the mean square energy fluctuation in a narrow frequency interval in a small segment of the string:

$$
\begin{aligned}
\langle\overline{\Delta E^2_{(a,\omega)}}\rangle &\equiv \langle\overline{\Delta H^2_{(a,\omega)}}(\{n_i\},\{n_i\})\rangle \\
&= \frac{1}{l^2}\sum_{j\neq k}\left(\hat{n}_j\hat{n}_k + \frac{1}{2}\left(\hat{n}_j + \hat{n}_k\right)\right)\hbar^2\omega_j\omega_k\frac{\sin^2\left((\omega_j-\omega_k)a\right)}{(\omega_j-\omega_k)^2}.
\end{aligned}
\tag{69}
$$

The right-hand side has exactly the same form as Eq. (53), except that the n's are replaced by $\hat{n}$'s. Eq. (60) tells us that $\hat{n}_j$ and $\hat{n}_k$ vary smoothly with j and k, so we can make the transition from sums to integrals in this case without any further assumptions. Proceeding in the exact same way as we did to get from Eq. (53) to Eq. (56), we arrive at:

$$
\langle\overline{\Delta E^2_{(a,\nu)}}\rangle = 2a\Delta\nu\left((\hat{n}_\nu h\nu)^2 + (\hat{n}_\nu h\nu)\nu\right). \tag{70}
$$

The thermal average of the mean excitation energy in a narrow frequency interval in a small segment of the string is given by (cf. Ch. 4, Eq. (39))

$$
\langle\overline{E_{(a,\nu)}}\rangle = \frac{a}{l}\left(\hat{n}_\nu h\nu\right)N_\nu = 2a\Delta\nu\left(\hat{n}_\nu h\nu\right). \tag{71}
$$

This is just Eq. (58) with $\hat{n}$ instead of n. With the help of this expression we can rewrite Eq. (70) as:

$$
\langle\overline{\Delta E^2_{(a,\nu)}}\rangle = \frac{\langle\overline{E_{(a,\nu)}}\rangle^2}{2a\Delta\nu} + \langle\overline{E_{(a,\nu)}}\rangle h\nu. \tag{72}
$$

This formula for the canonical-ensemble average of the mean square fluctuation of the energy in a narrow frequency range in a small segment of the string has exactly the same form as Eq. (59) for the mean square quantum uncertainty in the energy of this subsystem in an energy eigenstate of the system as a whole.

Our final result, Eq. (72), is the analogue of Einstein's famous 1909 formula for the mean square fluctuation of the energy in a narrow frequency range in a subvolume of a box with black-body radiation. That Eq. (72) emerges from the quantum-mechanical treatment of the modes of a string shows that the fluctuation formula, contrary to what Einstein thought, does not call for two separate mechanisms, one involving particles and one involving waves. In matrix mechanics, both terms arise from a unified dynamical framework. In the *Dreimännerarbeit* this unified mechanism is described in terms of quantized waves. If we focus on the occupation levels n_i rather than on the field $u(x, t)$, however, we see that the same mechanism can also be described in terms of particles, quanta of the field, satisfying Bose's statistics.

5 Assessment of the validity and the importance of Jordan's argument

The main conclusion we want to draw from our reconstruction of the fluctuation considerations in the *Dreimännerarbeit* is that they support the authors' claim—or rather Jordan's claim—that a straightforward application of the new matrix mechanics to a simple model of black-body radiation, viz. oscillations in a string fixed at both ends, leads to an expression for the mean square energy fluctuation in a narrow frequency range in a small segment of that string that has exactly the same form as the formula Einstein derived from statistical mechanics and Planck's black-body radiation law for the mean square energy fluctuation in a narrow frequency range in a subvolume of a box filled with black-body radiation. We also noted, however, that the authors use a sloppy notation and that the argument they present is incomplete.

At various points, the notation fails to reflect the crucial restriction to a narrow frequency range. We drew attention to a couple of passages in the text that clearly indicate that such a restriction is nonetheless in effect throughout the calculation. Since the entire derivation is for a finite frequency range, there are no problems with infinities (*pace* Ehlers, 2007, pp. 28–29). Another problem is that the authors do not distinguish in their notation between (in modern terms) operators and expectation values of operators in energy eigenstates. Here we have to keep in mind that this distinction had not fully crystalized when the paper was written. The authors had no clear notion yet of operators acting on states. They did not even have the general concept of a state (Duncan and Janssen, 2007, sec. 3).

In the absence of the general state concept, they did not distinguish between pure states and mixed states either. This did trip them up. The formula they derived is for the mean square *quantum uncertainty* in the energy of a subsystem in an energy eigenstate of the system as a whole, which is a *pure state*. What they should have derived to recover Einstein's fluctuation formula is a formula for the *thermal* mean square fluctuation in the energy of the subsystem, i.e., a canonical-ensemble average over energy eigenstates of the whole system, which is a *mixed state*. We showed in detail how to make this transition from quantum uncertainty to thermal fluctuations. Given the preliminary character of the theory he was working with, Jordan can be forgiven for the omission of this step in the *Dreimännerarbeit*, though he probably should have known better when he presented his result again in later publications. In Jordan's defense, we noted that Lorentz likewise omitted the corresponding step in the classical calculation.

With our admittedly not unimportant emendation, Jordan's result resolves Einstein's conundrum of the wave-particle duality of light, even though the treatment of such phenomena as the photoelectric effect and the Compton effect had to await the work of Dirac (1927), who developed the theory for the interaction between the quantized electromagnetic field and matter. As we saw in sec. 3.6, Jordan emphasized his resolution of the wave-particle conundrum in a number of publications. The main reason for the lukewarm reception of this result in the physics community of his day seems to have been that it looked suspicious because of the infinities one already encounters in this simple example of a quantum theory of free fields. This suspicion has lingered, even though, as we saw, Jordan managed to steer clear of infinities by focusing on a narrow frequency range.

The less than enthusiastic reaction of the physicists no doubt partly explains why Jordan's result has not become a staple of the historical literature on the wave-particle duality of light. Another factor responsible for its neglect in this context, as we suggested in sec. 3, may have been that Jordan's result was too many things at once. It was the resolution of the conundrum of the wave-particle duality of light but it was also a striking piece of evidence for matrix mechanics and a telltale sign that a quantum theory of fields was needed. Given how strongly Jordan felt about this last use of his result, it is perhaps only fitting that his derivation of Einstein's fluctuation formula has found its place in the historical literature not toward the end of histories of wave-particle duality but at the beginning of histories of quantum field theory. Still, the result only played a relatively minor role in the early stages of quantum mechanics and quantum field theory. By contrast, it is the denouement of the early history of the wave-particle duality of light. Regrettably, it has either been ignored in that context or doubts have been cast upon it. We hope that our paper will help remove those doubts so that Jordan's result can finally be given its rightful place in the heroic tale of Einstein, light quanta, and the wave-particle duality of light.

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