A service for the physicists?

B. L. van der Waerden's early contributions to quantum mechanics

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Dr. B. L. VAN DER WAERDEN, die benoemd is tot hoogleeraar in de faculteit der Wis- en Natuurkunde van de Rijks Universiteit te Groningen om onderwijs te geven in de elementaire wiskunde, de analytische, beschrijvende en hoogere meetkunde.

Outline

1. Biographical sketch
2. Spinor calculus - a calculus on demand (1929)
3. Modern representation theory: Groups with operators (1932)
4. Slater's method revisited (1932)
5. Conclusion

Bartel Leendert van der Waerden (1903–1996)

1919–1924	studies of mathematics and physics at Amsterdam Uni.
1924/25	studies at Göttingen Uni.
1925	military service in NL
March 1926	PhD-thesis, Amsterdam Uni.
1926/27	assistant lecturer at Hamburg Uni.
1927/28	assistant lecturer at Göttingen Uni.
Feb. 1927	habilitation at Göttingen Uni.
1928-1931	**professor at Groningen Uni. (NL)**
Summer 1929	**visiting professor at Göttingen Uni.**
1931-1945	**professor at Leipzig Uni.**
1946–1947	position with BPM (Shell), NL
1947	visiting lecturer, John Hopkins Uni. (Baltimore)
1948–51	professor at CWI/Amsterdam Uni.
1951–71	professor at Zürich Uni.

Van der Waerden’s contributions to physics

- Over Einstein’s relativiteitstheorie, *De socialistische gids* 6(1921), p.54–73, p.185–204
- **Spinoranalyse**, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1929, p.100–109
- ***Die gruppentheoretische Methode in der Quantenmechanik***, Springer, Berlin, 1932 (reprint Ann Arbor, Edward Brothers, 1944, engl. 1974)
- L. Infeld and B.L. van der Waerden, **Die Wellengleichung des Elektrons in der allgemeinen Relativitätstheorie**, *Sitzungsberichte der Preussischen Akademie der Wissenschaften* 7/10(1933), p.380–401 (Nachtrag 11/13, p.474)
- Die lange Reichweite der regelmäßigen Atomanordnung in Mischkristallen, *Zeitschrift für Physik* 118(1941), p.473–488
- Zur Quantentheorie der Wellenfelder, *Helvetica Physica Acta* 36(1963), p.945–962
- On measurements in quantum mechanics, *Zeitschrift für Physik* 190(1966), p.99–109
- *Mathematik für Naturwissenschaftler*, Bibliographisches Institut, Mannheim, 1975

P. Ehrenfest to B. L. van der Waerden, 8.10.1928:

I [Ehrenfest] would like to ask you [van der Waerden] about various mathematical things, very basic for you, because unfortunately a real group plague [Gruppenpest] has broken out in our physical journals. Almost all of my questions will refer to certain places in Weyl's new book: "Gruppentheorie und Quantenmechanik", and there mainly to the different "integer and half-integer" representations of the rotation group in three- and four-dimensional space.

[Ehrenfest Scientific Correspondence 10, S.6, 217, Museum Boerhaave, Leiden, translation MS]

Description of the concept of representation:

The task to find all “quantities” that are transformed linearly by the Lorentz-transformations according to any kind of rule, so that with a composition of two Lorentz-transformations the corresponding transformations of the “quantities” are combined too, i.e. so that the product of two Lorentz-transformations is expressed again by the product, is nothing else but the problem of the *representation* of the Lorentz-group through linear transformations.

[van der Waerden, 1929, p.101, emphasis in the original, translation MS]

Spinor calculus

Let $A \in SL(2,\mathbb{C}), (\xi_1, \xi_2) \in \mathbb{C}^2$.
Standard resp. complex conjugate standard representation of $SL(2,\mathbb{C})$:

$$A(\xi_1, \xi_2)^T \quad \text{resp.} \quad \overline{A}(\overline{\xi}_1, \overline{\xi}_2)^T$$

Generalization to (tensor-)products by factorwise operation of A resp. $\overline{A}$, e.g.:

$$\overline{\xi}_\lambda \overline{\eta}_\mu \zeta_\nu$$

($\lambda, \eta, \nu \in \{1,2\}$). Notation of quantities/spinors:

$$a_{\dot{\lambda}\dot{\mu}\nu}$$

Pulling down (pushing up) of - dotted - indices:

$$a^1 = -a_2, \quad a^2 = a_1$$

A mapping between bispinors $a_{\dot{\lambda}\mu}$ and $\mathbb{R}^4$ induces a $2:1$ homomorphism between $SL(2,\mathbb{C})$ and $\mathcal{L}_+^\uparrow$ ($\rightsquigarrow$ "two-valued" representation of $\mathcal{L}_+^\uparrow$)

[van der Waerden, 1929, p.101-103]

Relativistic wave equation for the electron (Weyl)

$$\boxed{\frac{1}{c}\left(\frac{h}{i}\frac{\partial}{\partial t}+\Phi_0\right)\psi+\sum_{r=1}^{3}s'_r\left(\frac{h}{i}\frac{\partial}{\partial x_r}+\Phi_r\right)\psi+mc\Gamma_0\psi=0}$$

where $\psi=(\psi_1,\psi_2,\psi_3,\psi_4),\psi_i:\mathbb{R}^4\longrightarrow\mathbb{C}$

and $s'_r\in Mat(4,\mathbb{C})$ with $s'_r=\begin{pmatrix} s_r & 0 \\ 0 & -s_r \end{pmatrix}\,(r=1,2,3)$

and s_r Pauli spin matrices:

$$s_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad s_2=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\quad s_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\Gamma_0=\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

with Φ_i a field, c speed of light, m mass and $h=\hbar$ the Planck constant.

[van der Waerden, 1929, p.106f.]

Wave equation in spinor form

$$-\left(\tfrac{h}{i}\partial^{\dot\lambda}_{\ \mu} + \Phi^{\dot\lambda}_{\ \mu}\right)\psi_{\dot\lambda} + mc\chi_\mu = 0$$
$$\left(\tfrac{h}{i}\partial_{\dot\mu}^{\ \lambda} + \Phi_{\dot\mu}^{\ \lambda}\right)\chi_\lambda + mc\psi_{\dot\mu} = 0,$$

where $\psi = (\psi_{\dot 1}, \psi_{\dot 2}, \chi_1, \chi_2)$
and $\psi_{\dot\lambda}, \chi_\mu : \mathbb{R}^4 \longrightarrow \mathbb{C}$,
with $\Phi^{\dot\lambda}_{\ \mu}$ a field, c speed of light, m mass and $h = \hbar$ the Planck constant.

[van der Waerden, 1929, p.107]

Groups with operators: history

1925	W. Krull, Über verallgemeinerte endliche Abelsche Gruppen
1926	W. Krull, Theorie und Anwendung der verallgemeinerten Abelschen Gruppen
1927	E. Noether, Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern
1928	O. Schmidt, Über unendliche Gruppen mit endlicher Kette
1929	E. Noether, Hyperkomplexe Größen und Darstellungstheorie
1930/31	B.L. van der Waerden, Moderne Algebra I, II

Definition: Group with operators

A set $\mathfrak{g}$ of *elements* $a, b, \ldots$ of any kind (e.g. of numbers, of linear transformations) is called a *group* if the following four conditions are satisfied:

(8.1.) A "product" $a \cdot b$ (or ab) is assigned to each pair of elements a, b in such a way that it belongs again to $\mathfrak{g}$.

(8.2.) The law of associativity $ab \cdot c = a \cdot bc$.

(8.3.) There exists a "unit element", e or 1, with the property $ae = ea = a$.

(8.4.) For each a of $\mathfrak{g}$ there exists an inverse a^{-1} in $\mathfrak{g}$, so that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ holds.

The group is called *abelian*, if $ab = ba$ always applies.

[. . .] **Generally one speaks of a *group with operators* if certain "multipliers" or "operators" θ with the property (8.5) [i.e.**

$$\theta(u + v) = \theta u + \theta v \qquad (u, v \in \mathfrak{g})]$$

are added to an abelian group.

[van der Waerden, 1932, S.28f, emphasis and translation MS]

Van der Waerden on Slater's method

However, there is a second method, in principle already older, recently applied successfully in particular by J.C. SLATER, which gets by with much simpler aids and which does not require the representation theory of the permutation group.

[van der Waerden, 1932, p.120, emphasis in original, translation MS]

Van der Waerden's table of configuration for three $2p-$electrons

			M_L	M_S
(2 1 1 +)	(2 1 0 +)	(2 1 -1 +)	0	$\frac{3}{2}$
(2 1 1 +)	(2 1 0 +)	(2 1 1 -)	2	$\frac{1}{2}$
(2 1 1 +)	(2 1 0 +)	(2 1 0 -)	1	$\frac{1}{2}$
(2 1 1 +)	(2 1 0 +)	(2 1 -1 -)	0	$\frac{1}{2}$
(2 1 1 +)	(2 1 -1 +)	(2 1 1 -)	1	$\frac{1}{2}$
(2 1 1 +)	(2 1 -1 +)	(2 1 0 -)	0	$\frac{1}{2}$
(2 1 1 +)	(2 1 -1 +)	(2 1 -1 -)	-1	$\frac{1}{2}$
(2 1 0 +)	(2 1 -1 +)	(2 1 1 -)	0	$\frac{1}{2}$
(2 1 0 +)	(2 1 -1 +)	(2 1 0 -)	-1	$\frac{1}{2}$
(2 1 0 +)	(2 1 -1 +)	(2 1 -1 -)	-2	$\frac{1}{2}$

[van der Waerden, 1932, p.121]

References

SLATER, JOHN CLARKE [1929], The theory of complex spectra, *Physical Review*, Bd. 34(10), S. 1293–1322, eingereicht am 7.6.1929.

VAN DER WAERDEN, BARTEL LEENDERT [1929], Spinoranalyse, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, Bd. ohne Angabe, S. 100–109.

VAN DER WAERDEN, BARTEL LEENDERT [1932], *Die gruppentheoretische Methode in der Quantenmechanik*, Julius Springer, Berlin, (Nachdruck durch Edward Brothers, Ann Arbor, 1944).

WEYL, HERMANN [1931], *Gruppentheorie und Quantenmechanik*, Wissenschaftliche Buchgesellschaft, Darmstadt, 2. Aufl., (unveränderter reprographischer Nachdruck der 2., umgearbeiteten Auflage, Leipzig 1931, erschienen 1977; 1. Auflage 1928).

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