Electron Spin

or

"klassisch nicht beschreibbare Art von Zweideutigkeit"

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Relativistic correction to Zeeman effect 1

- In his paper *Über den Einfluß der Geschwindigkeitsabhängigkeit der Elektronenmasse auf den Zeemaneffekt*, submitted in Dec. 1924, WOLFGANG PAULI observed that the velocity dependence of mass, $m = m_0/\sqrt{1-\beta^2}$, where $\beta = v/c$, effects the ratio of the (time-averaged, $\langle \cdots \rangle$) magnetic moment of a point charge q

$$\vec{M} = \tfrac{q}{2}\,\langle \vec{x} \times \vec{v} \rangle \tag{1}$$

to its (conserved) angular momentum

$$\vec{J} = m\,\vec{x} \times \vec{v} = m_0\,(\vec{x} \times \vec{v})/\sqrt{1-\beta^2} \tag{2}$$

- This so-called **gyromagnetic ratio** is hence given by

$$\frac{|\vec{M}|}{|\vec{J}|} = \frac{q}{2m_0}\,\left\langle \sqrt{1-\beta^2} \right\rangle =: \frac{q}{2m_0}\,\gamma \tag{3}$$

- For the single-electron Kepler problem one finds (Sommerfeld 1916), where $\mathcal{E}$ is the total energy, k the azimuthal quantum number, and $n = n_r + k$ the principal quantum number:

$$\gamma \;=\; 1 + \mathcal{E}/m_0c^2 \;=\; \left\{ 1 + \frac{\alpha^2 Z^2}{\left(n - k + \sqrt{k^2 - \alpha^2 Z^2}\,\right)^2} \right\}^{-1/2} \;\approx\; 1 - \frac{\alpha^2 Z^2}{2n^2} \tag{4}$$

Relativistic correction to Zeeman effect 2

- For higher Z one obtains significant deviations from the classical value $\gamma = 1$. For example, $Z = 80$ gives $g = 0.812$.
- In the core model, this relativistic correction modifies the assumed magneto-mechanical anomaly of the core **in a Z–dependent fashion**. For example, it leads to a suppression by a factor of $(2\gamma - 1)$ for the Zeeman splitting of the π-components, amounting to a decrease of about 18% for Mercury or Thallium.
- PAULI observed that this is definitely ruled out by spectroscopic data of RUNGE, PASCHEN, and BACK, whose experimental errors were estimated by LANDÉ to be less than 1%. PAULI summarises his findings:

> "If one wishes to keep the hypothesis that the magneto-mechanical anomaly is also based in closed electron groups and, in particular, the K shell, then it is not sufficient to assume a doubling of the ratio of the group's magnetic moment to its angular momentum relative to its classical value. In addition, one also needs to assume a compensation of the relativistic correction."

Pauli's new hypothesis

> "The closed electron configurations shall not contribute to the magnetic moment and angular momentum of the atom. In particular, for the alkalies, the angular momenta of, and energy changes suffered by, the atom in an external magnetic field shall be viewed exclusively as an effect of the radiating electron [Lichtelektron], which is also regarded as the place [der Sitz] of the magneto-mechanical anomaly. The doublet structure of the alkali spectra, as well as the violation of the Larmor theorem, is, according to this viewpoint, a result of a classically indescribable two-valuedness of the quantum-theoretic properties of the radiating electron."

- This seems to have been the first time that a quantum number was assigned without the obvious presence of a corresponding classical degree of freedom (whose existence was, at that stage, even flatly denied by PAULI).

Why classically indescribable?

- By the end of 1924 PAULI had arrived at a very strong scepticism concerning the usage of classical models in atomic physics. He e.g. speaks of *Modell-Vorurteilen* (letter to Sommerfeld 12/6/1924).
- Putting PAULI's general scepticism aside, what was his **specific** reason for claiming the classically indescribable nature of the new electron degree of freedom, even after GOUDSMIT's and UHLENBECK's 1925 note in *Die Naturwissenschaften*?
- The only explicit answer I know of makes the same point already made in a footnote in GOUDSMIT's and UHLENBECK's paper: that in order to achieve a gyromagnetic ratio of twice the size of that given by orbital motion, the equatorial parts of the electron's surface has to move at superluminal speeds ($\Leftarrow$ Lorentz's criticism; though not obvious whether $g = 2$ or other quantitative aspects).
- PAULI gives no details and seems to just rest on GOUDSMIT's and UHLENBECK'S remark, which in turn is based on ABRAHAM's electron theory of 1903 (see below).
- **Is that argument correct? That is, is a classical gyromagnetic factor $g = 2$ irreconcilable with Special Relativity (as is often read/heard)? More generally: What, at that time (1925), could rightfully been have said to quantitatively rule out classical electron models? (ignoring gavity)**

Quotes

"The electron must now assume the property (a) [a g-factor of 2], which LANDÉ attributed to the atom's core, and which is hitherto not understood. The quantitative details may well depend on the choice of model for the electron. [...] Note that upon quantisation of that rotational motion [of the spherical hollow electron], the equatorial velocity will greatly exceed the velocity of light."

UHLENBECK & GOUDSMIT 1925

"Emphasising the kinematical aspects one also speaks of the "rotating electron" (English "spin-electron"). However, we do not regard the conception of a rotating material structure to be essential, and it does not even recommend itself for reasons of superluminal velocities one then has to accept."

PAULI 1929

Addendum to his *"Allgemeine Grundlagen der Quantentheorie des Atombaues"*, in Müller-Pouillets Lehrbuch.

The naive classical electron: Electric field

- Consider a homogeneous charge distribution, ρ, of total charge Q on a sphere of radius R centred at the origin (we write $r := |\vec{x}|$ and $\vec{n} := \vec{x}/r$):

$$\rho(\vec{x}\,') = \frac{Q}{4\pi R^2}\,\delta(r' - R) \tag{5}$$

- It is the source for the scalar potential

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\,d^3x' = \frac{Q}{4\pi\varepsilon_0 R}\begin{cases} 1 & \text{for } r < R \\ R/r & \text{for } r > R \end{cases} \tag{6}$$

with corresponding electric field

$$\vec{E}(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0 r^2}\begin{cases} \vec{0} & \text{for } r < R \\ \vec{n} & \text{for } r > R \end{cases} \tag{7}$$

The naive classical electron: Magnetic field

- Let the charge distribution rotate rigidly with constant angular velocity $\vec{\omega}$. This gives rise to a current density

$$\vec{j}(\vec{x}\,') = (\vec{\omega} \times \vec{x}\,')\,\rho(\vec{x}\,') = \frac{Q}{4\pi R^2}\,(\vec{\omega} \times \vec{x}\,')\,\delta(r' - R) \tag{8}$$

- It is the source of a vector potential according to BIOT-SAVART's law:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\,d^3x = \frac{\mu_0 Q}{12\pi R}\,\vec{\omega} \times \begin{cases} \vec{x} & \text{for } r < R \\ \vec{x}\,(R/r)^3 & \text{for } r > R \end{cases} \tag{9}$$

with corresponding magnetic field (no addition to electric field)

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\begin{cases} 2\vec{M}/R^3 & \text{for } r < R \\ \big(3\vec{n}(\vec{n}\cdot\vec{M}) - \vec{M}\big)/r^3 & \text{for } r > R \end{cases} \tag{10}$$

- For $r > R$ this is a pure dipole field with dipole moment

$$\vec{M} := \tfrac{1}{3}QR^2\,\vec{\omega} \tag{11}$$

The naive classical electron: Energy distribution 1

- The energy of the electromagnetic field is given by

$$\mathcal{E} = \int_{\mathbb{R}^3} \tfrac{1}{2}\left(\varepsilon_0|\vec{E}|^2(\vec{x}) + \tfrac{1}{\mu_0}|\vec{B}|^2(\vec{x})\right)\, d^3x \tag{12}$$

- The electric and magnetic contributions are respectively given by

$$\mathcal{E}_e = \frac{Q^2}{8\pi\varepsilon_0\, R}\begin{cases} 0 & \text{from } r < R \\ 1 & \text{from } r > R \end{cases} \qquad \text{and} \qquad \mathcal{E}_m = \frac{\mu_0}{4\pi}\,|\vec{M}|^2/R^3 \begin{cases} 2/3 & \text{from } r < R \\ 1/3 & \text{from } r > R \end{cases} \tag{13}$$

- The total magnetic contribution can be written as

$$\mathcal{E}_m = \frac{\mu_0}{4\pi}\,|\vec{M}|^2/R^3 = \tfrac{1}{2}\, I\,|\vec{\omega}|^2\,, \qquad \text{where} \qquad I := \frac{\mu_0}{18\pi}\, Q^2 R \tag{14}$$

  may be called the **electromagnetic moment of inertia**. It has no mechanical interpretation in terms of a rigid rotation of the electrostatic energy distribution (see below)!

- Writing $m_e := \mathcal{E}_e/c^2$, one has

$$I = \tfrac{2}{3}\left(\tfrac{2}{3}\, m_e\, R^2\right) \tag{15}$$

The naive classical electron: Energy distribution 2

- The total electromagnetic energy can now be written as

$$\mathcal{E} = \mathcal{E}_e + \mathcal{E}_m = \frac{Q^2}{8\pi\varepsilon_0 R}\left\{1 + \tfrac{2}{9}\beta^2\right\} \tag{16}$$

  where we used $\varepsilon_0\mu_0 = 1/c^2$ and set $\beta := v/c$, where $v := R|\vec{\omega}|$.

- The ratio of magnetic ('kinetic') to total energy is then given by

$$\frac{\mathcal{E}_m}{\mathcal{E}} = \frac{\beta^2}{9/2 + \beta^2} \tag{17}$$

  which is a strictly monotonic function of β bounded above by 1 (as it should be).

- However, if we require $\beta < 1$, the upper bound is $2/11$.

The naive classical electron: Momentum distribution

- The momentum density of the electromagnetic field vanishes for $r < R$ and is given by

$$\vec{p}(\vec{x}) = \frac{\mu_0}{16\pi^2} Q\,(\vec{M} \times \vec{n})/r^5 \tag{18}$$

  for $r > R$ ($1/c^2$ times 'Poynting vector').
- The angular-momentum density also vanishes for $r < R$. For $r > R$ it is given by

$$\vec{\ell}(\vec{x}) = \vec{x} \times \vec{p}(\vec{x}) = \frac{\mu_0}{16\pi^2} Q\,\frac{\vec{M} - \vec{n}(\vec{n} \cdot \vec{M})}{r^4} \tag{19}$$

- Hence the total linear momentum vanishes, whereas the total angular momentum is given by

$$\vec{J} := \int_{r>R} \vec{\ell}(\vec{x})\, d^3x = I\vec{\omega} \tag{20}$$

  with the **same** I (moment of inertia) as in the energy expression (14).

The naive classical electron: Gyromagnetic ratio

- The gyromagnetic ratio now follows from expressions (11) for $\vec{M}$ and (20) for $\vec{J}$:

$$\frac{|\vec{M}|}{|\vec{J}|} = \frac{6\pi\, R}{\mu_0\, Q} =: g\, \frac{Q}{2m} \tag{21}$$

where m denotes the total mass, which is here given by

$$m := \mathcal{E}/c^2 = \frac{\mu_0}{8\pi}\frac{Q^2}{R}\left\{1 + \tfrac{2}{9}\beta^2\right\} \tag{22}$$

- Hence g can be solved for:

$$g = \frac{3}{2}\left\{1 + \tfrac{2}{9}\beta^2\right\} \tag{23}$$

so that

$$\boxed{\tfrac{3}{2} < g < \tfrac{11}{6} \qquad \text{if} \qquad 0 < \beta < 1} \tag{24}$$

- Note that $g = 2 \Leftrightarrow 1 + \frac{2}{9}\beta^2 = \frac{4}{3} \Leftrightarrow m = \frac{4}{3}m_e$, which is ABRAHAM's value for the translatory mass. This was used by GOUDSMIT and UHLENBECK to suggest ABRAHAM's theory predicted $g = 2$.

Intermezzo: Kinematics of Faraday lines 1

> "If one wishes to represent these lines of force as something material in the usual sense, one is tempted to interpret dynamical processes [of the em. field] as motions of these lines of force, so that each such line can be followed in time. It is, however, well known that such an interpretation leads to contradictions.
> In general we have to say that it is possible to envisage extended physical objects to which the notion of motion [in space] does not apply"
>
> A. EINSTEIN: Äther und Relativitätstheorie, 1920

Intermezzo: Kinematics of Faraday lines 2

- The electrostatic energy density $\rho_e(\vec{x})$ corresponds to a mass density

$$\rho_m(\vec{x}) := \rho_e(\vec{x})/c^2 = \left(\frac{\mu_0}{32\pi^2}\right)\frac{Q^2}{r^4} \tag{25}$$

- If this energy is thought of as being attached to **material** lines of force, to which the usual kinematical concept of motion applies, the following moment of inertia for the shell $R < r < R'$ would result:

$$I(R') = \int_{R<r<R'} \rho_m(\vec{x})\,(r\sin\theta)^2\, d^3x = \left(\frac{2\mu_0}{27\pi}\right)\, Q^2\,(R' - R) \tag{26}$$

- This diverges for $R' \to \infty$!

A less naive model of the electron: Including Poincaré stresses

- The previous model is now modified in the following three aspects (COHEN & MUSTAFA 1986):

  1. The infinitesimally thin spherical shell is given a small rest-mass of constant surface density $m_0/4\pi R^2$.
  2. Poincaré stresses are taken into account.
  3. The rotational velocity is small, so that $(v/c)^n$ terms are neglected for $n \geq 2$.

- The energy-momentum tensor has now three contributions, corresponding to the material shell, the Poincaré stresses, and the electromagnetic field:

$$\mathbf{T} \;=\; \frac{m_0}{4\pi R^2}\,\delta(r-R)\,\mathbf{u}\otimes\mathbf{u} \;-\; \underbrace{\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q^2}{16\pi R^3}}_{\text{surface tension}}\,\delta(r-R)\,\mathbf{P} \;+\; \mathbf{T}_{em} \tag{27}$$

where

$$\mathbf{u} = \partial_t + \omega\,\partial_\varphi \qquad \text{and} \qquad \mathbf{P} = \text{projector onto } \{u, \partial_r\}^\perp \tag{28}$$

Angular momentum

- The angular momentum resulting from $\mathbf{T}$ receives contributions from the shell, the electromagnetic field (already calculated), and the Poincaré stress (again we set $m_e := \mathcal{E}_e/c^2$):

$$J = -\frac{1}{c^2}\int_{\mathbb{R}^3} \partial_t \cdot \mathbf{T} \cdot \partial_\varphi \, d^3x = \Big(m_0 + \tfrac{2}{3}m_e - \underbrace{\frac{\mu_0}{4\pi}\frac{Q^2}{4R}}_{m_e/2}\Big)\,\tfrac{2}{3}\omega R^2 = \big(m_0 + \tfrac{1}{6}m_e\big)\,\tfrac{2}{3}\omega R^2 \qquad (29)$$

- Note that the effect of Poincaré stress is to **diminish** the electromagnetic angular momentum **by a factor of $\mathbf{1/4}$** ($\tfrac{2}{3}m_e \rightarrow \tfrac{1}{6}m_e$). This will give rise to an **enhancement** of the g-factor since the magnetic moment remains unchanged.
- To linear order in ω the kinetic energy does not contribute to the overall mass, m, which is hence given by $m = m_0 + m_e$. Therefore

$$J = \left(1 + 5\,\frac{m_0}{m}\right)\,\frac{m\omega R^2}{9} \qquad (30)$$

Magnetic moment and g-factor

- The magnetic moment is the one already calculated:

$$M = \frac{1}{3} Q R^2 \omega \tag{31}$$

so that the g-factor follows

$$g = \frac{2m}{Q}\frac{M}{J} = \frac{6}{1 + 5\, m_0/m} \tag{32}$$

- **g ranges between 1 and 6**, corresponding to $m = m_0$ (i.e. no electromagnetic contribution) and $m_0 = 0$ (i.e. no shell contribution) respectively. Note that the latter case gives 4 times the naive value, 3/2, as already anticipated above.

- Compare the measured values for g for electron and proton:

$$g_{\text{electron}} = 2.0023193043622 \quad \text{and} \quad g_{\text{proton}} = 5.585694713 \tag{33}$$

- So what's the snag?

Constraints by slow-rotation approximation

- The independent microscopic parameters were

$$P = \big(m_0\,,\;Q\,,\;R\,,\;\omega\big)$$

  which uniquely determine the independent physical observables,

$$O = \big(m\,,\;Q\,,\;g\,,\;J\big) = O(P)$$

- The function $O = O(P)$ can be inverted, $P = P(O)$, leading in particular to

$$\frac{\omega R}{c} = \left[\frac{Q^2}{4\pi\varepsilon_0\hbar c}\right]^{-1}\left[\frac{2J}{\hbar}\right]\left[\frac{9(g-1)}{5}\right] = \left[\frac{1}{\alpha}\right]\left[\frac{n_J}{n_Q^2}\right]\left[\frac{9(g-1)}{5}\right] \tag{34}$$

  where we set $J = n_j\,\frac{1}{2}\hbar$, $Q = n_Q\,e$, and $\alpha := e^2/4\pi\varepsilon_0\hbar c \approx 1/137$ is the fine-structure constant.
- The condition $\omega R/c \ll 1$ implies $n_Q \gg 16$. Hence the slow-rotation approximation does not apply to small (in units of e) charges.
- **A fully special-relativistic calculation needs to be done!**

Constraints by energy-dominance

- The functional dependencies $P(O)$ give the following relations among the masses:

$$m_0 = m\,\frac{6-g}{5g}\,, \quad m_e := \frac{\mu_0}{4\pi}\frac{Q^2}{2R} = m - m_0 = m\,\frac{6(g-1)}{5g} \tag{35}$$

- The Poincare-stress-part of $\mathbf{T}$ can be written as

$$-\frac{1}{2}\,\frac{m_e}{4\pi R^2}\,c^2\,\delta(r-R)\,\mathbf{P} \tag{36}$$

- The ratio between the modulus of the Poincaré-stress and the rest-energy of the 'material' energy-momentum tensor is hence given by

$$\frac{|\mathbf{T}(e_\theta, e_\theta)|}{\mathbf{T}(u,u)} = \frac{m_e}{2m_0} = \frac{3(g-1)}{6-g} \quad \leq 1 \quad \Longleftrightarrow \quad g \leq \frac{9}{4} \tag{37}$$

SUMMARY

- $\mathbf{g = 2}$ **as such does not contradict Special Relativity.**
- **Models for low charges cannot be treated in slow-rotation approximation.**
- **A fully special-relativistic treatment has, to my knowledge, never been carried through.**

Thank You!