Weyl entering the ‚new' quantum mechanics discourse

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1. **Intro: Weyl in 1925 and correspondence Weyl –Born – Jordan 11/1925**

2. **From commutation rules to *Abelian ray representations***

3. **Weyl's answer to the quantization problem**

4. **Outlook: reception and repercussions**

1. Intro: Weyl in 1925 …

- Weyl in 1924 and early 1925 : Work on representation theory:

  ``Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen. I, II, III und Nachtrag." *Mathematische Zeitschrift* **23:** 271--309, **24**: 328----395, 789--791. GA II, 543—646

- Finished in April 1925. Then work on „Philosophie der Mathematik und Naturwissenschaften“ (published 1926/1927)

- Two reasons, why Weyl thought group representations to be important for physics --- already in the context of GRT: (I) role of tensors in differential geometry because of irreducible representations of SL(n,R) all in tensor products from natural (fundamental) representation– importance of symmetry conditions
  (II) analysis of the problem of space (algebraic part) could be answered using group representations (E. Cartan)

- September – November correspondence with Max Born and Pascual Jordan on new quantum mechanics

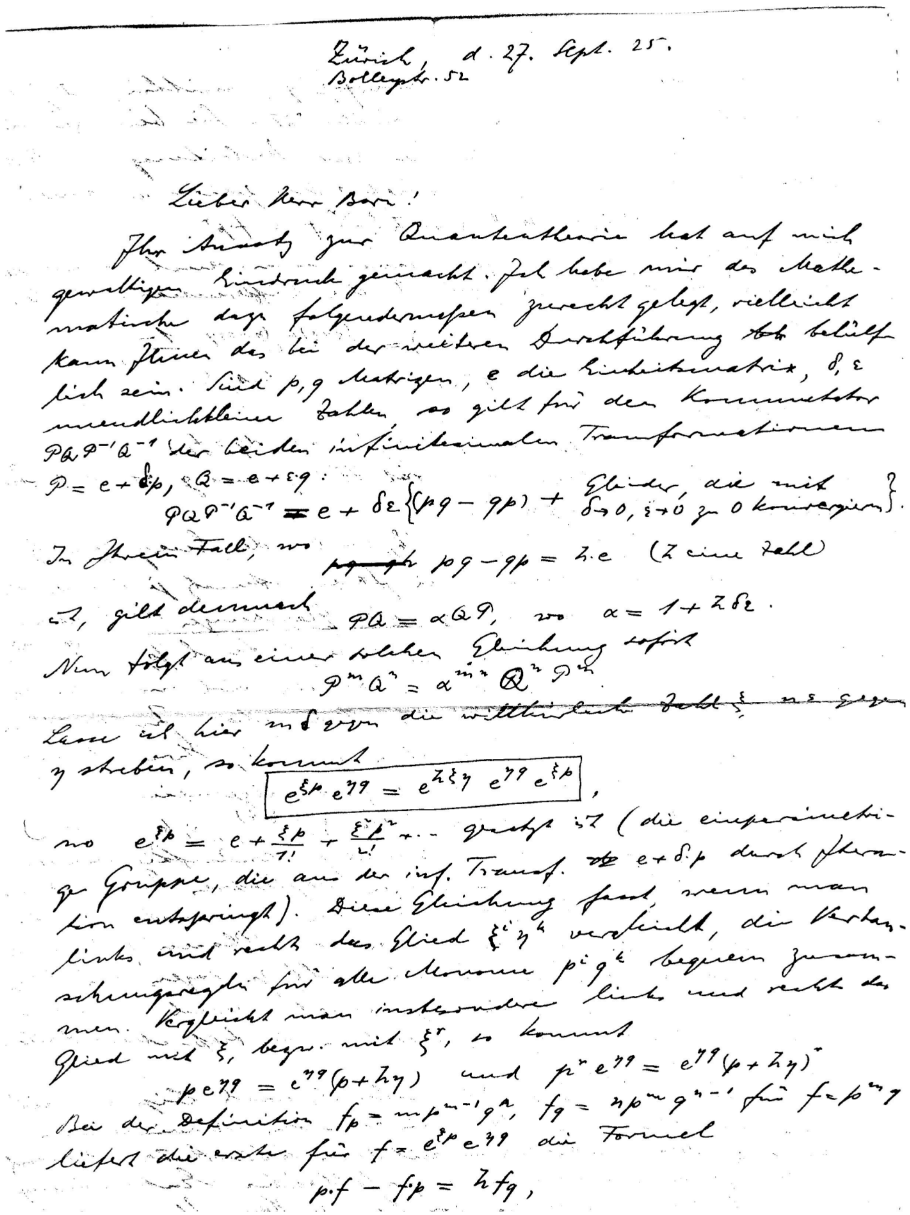
Weyl to Born, Zürich	27 Sep. 1925 (Staatsbibliothek Berlin, also in AHQO – NL Jordan (!) 965)
Born to Weyl, Göttingen	03 Oct. 1925 (ETH Hs 91: 488)
Weyl to Jordan, Zü	13 Oct. 1925 (StB Berlin dito, NL Jordan 638 + AHQP)
Jordan to Weyl, Gö	xx. Nov. 1925 (ETH Hs 91: 626)
Weyl to Jordan, Zü (postcard)	23. Nov. 1925 (StB Berlin dito, NL Jordan 638 + AHQP)
Weyl to Jordan, Zü (postcard)	25. Nov. 1925 (StB Berlin dito, NL Jordan 638 + AHQP)

Weyl to Born, 27 Sep. 1925:

„Ihr Ansatz zur Quantentheorie hat auf mich gewaltigen Eindruck gemacht. Ich habe mir das Mathematische dazu folgendermaßen zurecht gelegt, vielleicht kann Ihnen das bei ´der weiteren Durchführung behülflich sein ...“

„ Dear Herr Born,
Your Ansatz to QM has deeply impressed me. I have figured out the mathematical Side for myself, perhaps it may be useful For your further progress …“

(Born had visited Zürich in September 1925 and had informed Weyl on the recent progress made in Göttingen B-H-J)



Zürich, d. 27. Sept. 25.
Bolleystr. 52

Lieber Herr Born!

Ihr Ansatz zur Quantentheorie hat auf mich gewaltigen Eindruck gemacht. Ich habe mir das Mathematische dazu folgendermaßen zurecht gelegt, vielleicht kann Ihnen das bei der weiteren Durchführung behülflich sein. Sind p, q Matrizen, e die Einheitsmatrix, δ, ε unendlichkleine Zahlen, so gilt für den Kommutator $PQP^{-1}Q^{-1}$ der beiden infinitesimalen Transformationen $P = e + \delta p$, $Q = e + \varepsilon q$:

$$PQP^{-1}Q^{-1} = e + \delta\varepsilon\{(pq - qp) + \text{Glieder, die mit } \delta \to 0, \varepsilon \to 0 \text{ zu } 0 \text{ konvergieren}\}.$$

In Ihrem Fall, wo $pq - qp = z \cdot e$ (z eine Zahl) ist, gilt demnach $PQ = \alpha QP$, wo $\alpha = 1 + z\delta\varepsilon$.

Nun folgt aus einer solchen Gleichung sofort

$$P^m Q^n = \alpha^{mn} Q^n P^m.$$

Lässt man hier $m\delta$ gegen die willkürliche Zahl ξ, $n\varepsilon$ gegen η streben, so kommt

$$\boxed{e^{\xi p} e^{\eta q} = e^{z\xi\eta} e^{\eta q} e^{\xi p}},$$

wo $e^{\xi p} = e + \frac{\xi p}{1!} + \frac{\xi^2 p^2}{2!} + \dots$ gesetzt ist (die einparametrige Gruppe, die aus der inf. Transf. $e + \delta p$ durch Iteration entspringt). Diese Gleichung fasst, wenn man links und rechts das Glied $\xi^r \eta^s$ vergleicht, die Vertauschungsregeln für alle Monome $p^r q^s$ bequem zusammen. Vergleicht man insbesondere links und rechts das Glied mit ξ, bezw. mit ξ^r, so kommt

$$p e^{\eta q} = e^{\eta q}(p + z\eta) \quad \text{und} \quad p^r e^{\eta q} = e^{\eta q}(p + z\eta)^r$$

Bei der Definition $f_p = m p^{m-1} q^n$, $f_q = n p^m q^{n-1}$ für $f = p^m q^n$ liefert die erste für $f = e^{\xi p} e^{\eta q}$ die Formel

$$p f - f p = z f_q,$$

2. _From commutation rules to *Abelian ray representations*

For Born's matrices p,q, considered as infinitesimal generators, Weyl considered the corresponding 1- parameter groups P(s), Q(t)

$$P(s) = 1 + s\,p + \dots \qquad Q(t) = 1 + t\,q + \dots \quad (s, t \in \mathbb{R})$$

The Heisenberg commutation rule

$$pq - qp = \hbar 1, \tag{1}$$

with $\hbar$ “a number”, was/is equivalent to

$$PQ = \alpha QP, \tag{2}$$

with $\alpha(s,t)$ complex factor.
For $|\alpha| = 1$ this is just a phase factor,

„… which one could deny a physical meaning“ (Weyl to Born, 27 Sep 1925)
(2) Weyl‘s *quasi-commutation rule* [my expression, E.S.]

Abelian ray representations: non-trivial commutators (but simple)

For Hermitian matrix A Weyl took the corresponding anti-Hermitian C:= i A and the generated 1-parameter unitary group U(s):

$$U(s) = e^{isA} = e^{sC}\ , \quad s \in \mathbb{R}$$

For k such matrices $C_1, \ldots, C_k$, generating an abelian group G

$$U(s_1, \ldots, s_k) = e^{i \sum_{\nu} s_\nu C_\nu}\ , \quad s \in \mathbb{R}$$

In QM commutation of the C-s may be weakened

$$C_j C_k - C_k C_l = i c_{il} \cdot 1 \qquad (3)$$

with skew-symmetric coefficients (c_{ij}) (*commutator form*). For an irreducible group, the commutator form is non-degenerate ($|c_{ij}| \neq 0$). It can be normalized by change of generators to matrix blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Canonical basis for infinitesimal generators of G

New generators can be written as

$$iP_\nu\ , \quad iQ_\nu \quad (\nu = 1, \dots n) \quad P_\nu, Q_\nu$$

with P_ν, Q_ν Hermitian and

$$i(P_\nu Q_\nu - Q_\nu)P_\nu = c \cdot 1\ , \quad c = 1,\ \hbar \dots \tag{4}$$

All other commutators $= 0$.

Weyl: $P_1, \dots P_k, Q_1, \dots Q_k$ is *canonical basis* for G.

For

$$A(s) = e^{i\sum s_\nu P_\nu}\ , \quad B(s) = e^{i\sum t_\nu Q_\nu}$$

$$W(s,t) := A(s)B(t)$$

the commutation relations in G acquire the form

$$A(s)B(t) = e^{ic\sum \nu s_\nu t_\nu} B(t)A(s) \tag{5}$$

Abelian ray representations

The commutative addition for (s,t), (s',t') in $\mathbb{R}^n$ reappears here slightly deformed as:

$$W(s+s',t+t') = e^{-ic<s,t>} W(s,t)W(s',t') \,, \qquad (6)$$

where $\quad <s,t> := \sum_{\nu} s_\nu t_\nu$.

Weyl called the 'slightly deformed' representation

$$\mathbb{R}^n - -- \longrightarrow G$$

an *irreducible group of abelian ray rotations.*

Later terminology:
projective (or ray) representation of $\mathbb{R}^n$

Abelian ray representations characterize quantum „kinematics"

Weyl had found a structural reason,

founded in group theory,

for the canonical pairing of the P, Q operators satisfying Heisenberg commutation (4).

The latter arose as *infinitesimal counterparts* of the integral *quasi-commutation* rule (5) for the case of generators.

$$e^{isP_\nu}e^{itQ_\nu} = e^{i\hbar st}e^{itQ_\nu}e^{isP_\nu}$$

He concluded (a little later in the paper):

„The kinematical character of a physical system is expressed by an irreducible Abelian rotation group, the substrate of which [i.e. the set on which it operates, E.S.] **is the ray field (Strahlenkörper) of the ‚pure cases'." (Weyl 1927, 118)**

3. Weyl‘s approach to the quantization problem

Already in the letter to Jordan 23 Nov 1925 Weyl indicated that

„... the domain of reasonable functions H has to be characterized by the Ansatz

$$\int e^{\xi p+\eta q}\,\varphi(\xi,\eta)\,d\xi d\eta$$

This is less formal than $\sum p^m q^n$ **. ...“**

(Warning: imaginary **i** omitted in the exponential. Apparently **Fourier transform** meant)

In his publication 1927 Weyl argued that a classical mechanical system with n degrees of freedom and momentum resp. location parameters

$$p_1,\ldots,p_n,q_1,\ldots q_n$$

corresponds to a quantum kinematical system with canonical basis of generators

$$iP_\nu,\ iQ_\nu \quad (1\le\nu\le n)$$

Quantum system first, classical counterpart. Transfer of observables from 2nd to 1st

For classical *physical quantities* given by *functional expressions* $f(p,q)$.

"...it remains to be seen how such an expression might be transferred to matrices ..."

Problem of noncummutativity:

$$p^2q \mapsto QP^2, \quad \text{or} \quad PQP \quad \text{or} \quad P^2Q$$

Weyl's answer: Fourier transform $\xi(s,t)$

$$f(p,q) = \int e^{i(ps+qt)}\xi(s,t)\, dsdt \tag{7}$$

of $f(p,q)$'s Fourier inverse

$$\xi(s,t) = \left(\frac{1}{2\pi}\right)^n \int e^{i(ps+qt)} f(p,q)\, dpdq$$

suggests to use the operator integral

$$\mathcal{F}(f) := F = \int e^{i(Ps+Qt)}\xi(s,t)\, dsdt \tag{8}$$

Two types of composition for observables (Weyl 1927)

(i) Composition of classical *physical quantities*,
here real functions on $\mathbb{R}^n$

$$f(p,q), \quad g(p,q)$$

$$f \cdot g$$

(ii) Composition of *Weyl quantized observables*

$$\mathcal{F}(f) \; = \; \int e^{i(Ps+Qt)} \xi(s,t)\, dsdt$$
$$\mathcal{F}(g) \; = \; \int e^{i(Ps+Qt)} \eta(s,t)\, dsdt; ,$$
$$(\quad \eta = \hat{g})$$
$$\mathcal{F}(f) \; \circ \; \mathcal{F}(g)$$

(i) commutative, (ii) noncummutative.

Weyl did not look at algebraic ramifications of the modified product structure.