A service for the physicists? B. L. van der Waerden's early contributions to quantum mechanics*

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Bartel Leendert van der Waerden (1903-1996) was a scientist with a wide range of interests. He contributed to invariant theory, algebraic geometry, algebra, topology, statistics and probability theory, as well as to physics and to the history of mathematics, astronomy and physics.[1] In this paper I will try to characterize his early contributions to quantum mechanics which he wrote around 1930.[2] All of these deal with group theory, a mathematical theory which was well established at the time but whose application to quantum mechanics was quite controversial. How did van der Waerden come to publish something in this field? What kind of mathematics did he use? What was his stance in the debate about the "group plague"?

In order to give an idea in which direction these questions might be answered three examples from his papers will be discussed. These examples concern van der Waerden's development of the spinor calculus, his introduction to representation theory by a concept called group with operators, and his treatment of Slater's method, a method which explicitly was aimed at avoiding group theory. Local networks (in Göttingen, the Netherlands and Leipzig) which were a stimulus for van der Waerden's work in physics and, to some extent, had an influence on the direction of his research are sketched. By comparing van der Waerden's approach to that of H. Weyl and E. Wigner, two main advocates of the group-theoretic method in quantum mechanics at the time, some of its characteristic features are revealed and the wider (scientific) context is brought

*I dedicate this article to Erhard Scholz on the occasion of his 60th birthday. A more detailed analysis of van der Waerden's early contributions to quantum mechanics will be given in my PhD thesis "Die physikalischen Beiträge des jungen Bartel Leendert van Waerden" (Wuppertal, to appear 2008).

[1] For a list of his publications see Top and Walling [1994]. Despite its impressive length it is still incomplete.

[2] van der Waerden [1929, 1932]; Infeld and van der Waerden [1933].

into the analysis. Finally, an attempt is made to answer the question of the title with respect to the discussed examples.

Background

Van der Waerden[3] studied mathematics and physics with H. de Vries, G. Mannoury, R. Weitzenböck, L. E. J. Brouwer and J. D. van der Waals jr. at Amsterdam university between 1919 and 1924. In autumn 1924 he went to Göttingen on a Rockefeller grant. There he studied for one year with E. Noether, R. Courant and D. Hilbert. During this stay van der Waerden got to know the new methods of modern algebra developed by Noether and E. Artin and others and which he applied to give a better foundation of the Schubert calculus in his thesis. Although van der Waerden had done physics in Amsterdam, according to his own account it was in Göttingen where he studied mathematical physics in 1924 and read Courant and Hilbert's recent book on the methods of mathematical physics, and was deeply impressed by it.[4] Brouwer had given him a letter of recommendation for Courant and van der Waerden was soon drawn into Courant's small working unit on mathematical physics consisting of K. O. Friedrichs, H. Lewy and P. Jordan. Van der Waerden left Göttingen in 1925 to do his military service in the Netherlands and to write his PhD thesis. Then, in summer 1926, he went to Hamburg on the rest of his Rockefeller grant where he continued to study modern algebra with Artin. When the grant ran out he got a position as assistant to H. Blaschke in Hamburg. So, van der Waerden was not in Göttingen, but in the Netherlands and in Hamburg when W. Heisenberg, M. Born and Jordan developed matrix mechanics.

Figure 1: Bartel Leendert van der Waerden, Groningen 1928

In spring 1927 he returned to Göttingen where he worked as an assistant for Courant. He wrote his habilitation on Bézout's theorem. In 1928 he got his first professorship in Groningen in the Netherlands. As we will see, his time in Groningen was quite important for the development of his first work on quantum theory. He returned as a visiting professor to Göttingen in summer 1929. In 1930/31 he published a text book on modern algebra which was based on the lectures of Noether and Artin. It quickly became a bestseller. In May,

[3] For information on van der Waerden's biography see Gross [1973]; Eisenreich [1981]; van der Waerden [1983]; Frei [1993]; Frei et al. [1994]; Dold-Samplonius [1994]; Scriba [1996a,b]; Dold-Samplonius [1997]; Frei [1998]; Thiele [2004]; Soifer [2004a,b, 2005]. There is no scientific biography to date. A first step in this direction is a paper by Schappacher [2003] on van der Waerden's early contributions to the development of algebraic geometry. The photo of van der Waerden (Figure 1) is taken from the photo collection Veenhuijzen, Centraal Bureau voor Genealogie (Den Haag).

[4] Courant and Hilbert [1924].

1931, van der Waerden was appointed professor in Leipzig. He was especially looking forward to going there because Heisenberg and F. Hund were there too.[5] In fact, he regularly attended their seminar on the structure of matter. In the winter term 1931/32 he gave a course of lectures on group-theoretic methods in quantum mechanics. He also proposed and supervised the PhD thesis of H. A. Jahn on the rotation and oscillation of the methane molecule which included the Jahn-Teller effect.[6] It was in Groningen and Leipzig where he wrote the papers on quantum mechanics from which the examples for this analysis are taken.

Van der Waerden published several works on physics.[7] His first publication at the age of 18 was a popular account of special and general relativity theory based on a lecture given by P. Ehrenfest.[8] Several years later he published three works on quantum mechanics, one along with L. Infeld. Van der Waerden developed a spinor calculus in special and general relativity theory and applied it to the wave equation of the electron[9] and he wrote a monograph on the group-theoretic method in quantum mechanics[10]. This monograph was the last in a row of three books on the same subject. In 1931 both Wigner and Weyl had published monographs, too.[11] In the case of Weyl, this was the second, revised edition of his comprehensive book from 1928. When van der Waerden learned of Wigner's publication he almost withdrew from his project. It was Courant who convinced him to carry on.[12]

Group theory had been introduced into quantum mechanics in 1926/27 by Heisenberg, Wigner, J. von Neumann and Weyl.[13] With the help of this new method one was able to mathematically deduce the quantum numbers (except for the quantum number n) which had been based on an empirical analysis of spectra of atoms and molecules. The algebraically simple structures of irreducible representations (of the group of rotations and of the group of permutations) could be directly related to quantum numbers. Moreover, Wigner and Weyl also explored the conceptional power of group theory for the foundation of quantum mechanics. W. Heitler and F. London applied group-theoretic methods to explain the binding of atoms. The method, however, was met with resistance by a lot of physicists and chemists, mainly because they were not acquainted with group theory and found it difficult to learn. There was also a feeling that group-theoretic reasoning was essentially "not physical" (nicht physikalisch).[14] The term "group plague" (Gruppenpest) was coined. It was within this context that van der Waerden entered quantum mechanics.

[5]For information on the physics department in Leipzig during Heisenberg's professorship see e. g. Kleint and Wiemers [1993], on the interrelationship between mathematics and physics in Leipzig see Schlote [2007].

[6]Jahn [1935].

[7]van der Waerden [1921, 1929, 1932]; Infeld and van der Waerden [1933]; van der Waerden [1941, 1963, 1966, 1975]. He also contributed to the history of quantum mechanics [van der Waerden, 1960, 1967, 1973, 1976].

[8]van der Waerden [1921].

[9]van der Waerden [1929]; Infeld and van der Waerden [1933].

[10]van der Waerden [1932].

[11]Wigner [1931]; Weyl [1931].

[12]I thank Volker Remmert for this piece of information.

[13]For information on the history of the group-theoretic method in quantum mechanics see Mackey [1988a,b]; Sigurdsson [1991], Mehra and Rechenberg [chap. III.4(e) 2000], Chayut [2001]; Scholz [2006] and the contributions of E. Scholz and C. Smeenk in this volume.

[14]Wigner [1931, preface, p.V].

Spinor calculus - a calculus on demand

Van der Waerden developed spinor calculus in Groningen. He did so at the request of Ehrenfest in Leiden in spring 1929. Ehrenfest's question dates back to autumn 1928 when he tried to get a grip on group-theoretic methods in order to understand the works of Weyl, Wigner and von Neumann. Ehrenfest organized a series of lectures inviting specialists like Wigner, W. Pauli, Heitler, London and von Neumann. He also invited van der Waerden who had just got his first professorship in Groningen. Ehrenfest used van der Waerden as a kind of mathematical advisor:

> I [Ehrenfest] would like to ask you [van der Waerden] about various mathematical things, very basic for you, because unfortunately a real group plague [Gruppenpest] has broken out in our physical journals. Almost all of my questions will refer to certain places in Weyl's new book: "Gruppentheorie und Quantenmechanik", and in it mainly to the different "integer and half-integer" representations of the rotation group in three- and four-dimensional space.[15] [Ehrenfest to van der Waerden, 8.10.1928]

Van der Waerden was drawn into a Dutch circle of physicists by Ehrenfest, a circle consisting mainly of Ehrenfest's students and former students like H. Kramers, G. Uhlenbeck, S. Goudsmit, D. Coster and H. B. G. Casimir.

Van der Waerden developed a formalism to handle "spinors," a term probably coined by Ehrenfest. Spinors are quantities in representation spaces of the Lorentz group or of its subgroups. They had appeared implicitly or explicitly a couple of times in quantum mechanics: in the works of Pauli, Wigner, Weyl and P. A. M. Dirac. Van der Waerden was asked by Ehrenfest to develop a spinor calculus modelled on tensor calculus to handle these quantities more easily. So Ehrenfest had a mathematical formalism designed for calculation in mind.

In his article van der Waerden [1929] developed this calculus with a minimum of mathematical prerequisites: He neither touched the theory of representation, nor did he explain the theory of invariants or the underlying geometric picture. For example, he did not give a formal, let alone axiomatic definition of representation, but rather mentioned it, in passing by, in a concrete setting:

> The task to find all "quantities" that are transformed linearly by the Lorentz-transformations according to any kind of rule, so that with a composition of two Lorentz-transformations the corresponding transformations of the "quantities" are combined too, i. e. so that the product of two Lorentz-transformations is expressed again

[15] "Ich [Ehrenfest] habe Sie [van der Waerden] ueber verschiedene fuer Sie ganz elementare mat[h]ematische Dinge zu fragen, da ja leider eine wahre Gruppenpest in unseren physikalischen Zeitschriften ausgebrochen ist. Fast alle meine Fragen werden sich auf bestimmte Stellen aus dem neuen Buch von Weyl: Gruppentheorie und Quantenmechanik beziehen und da wieder hauptsaechlich auf die verschiedenen ‚ganz- und halbzahligen' Darstellungen der Drehgruppe im Drei- und vierdimensionalen Raum." [Museum Boerhaave in Leiden (MB), Ehrenfest Scientific Correspondence (ESC) 10, S.6, 217] The translation as well as the other translations in this paper are mine (MS).

> by the product, is simply the problem of the *representation* of the Lorentz group through linear transformations.[16] [van der Waerden, 1929, p.101, emphasis in the original]

This was not in line with a modern approach to representation theory of that time. It shows van der Waerden's capability and willingness to adapt to different scientific contexts and audiences.

Van der Waerden introduced the spinor formalism by letting the special linear group of complex 2×2 matrices ($SL_2\mathbb{C}$) act on a two-dimensional complex vector space. He described the standard representation and the complex conjugate standard representation of $SL_2\mathbb{C}$. He did this by explicitly giving the equations for the transformed vectors. He then generalized the operation of $SL_2\mathbb{C}$ to factors of "products" consisting of components of vectors in a two-dimensional space by factorwise operation, i.e. he described the action of $SL_2\mathbb{C}$ on tensor products. He introduced the following notation: If $\overline{\xi}_\lambda$ and $\overline{\eta}_\mu$ are transformed by the complex conjugate standard representation and ζ_ν by the standard representation then he denoted quantities transforming like the "product"

$$\overline{\xi}_\lambda \overline{\eta}_\mu \zeta_\nu$$

of these components by the spinor

$$a_{\dot{\lambda}\dot{\mu}\nu}$$

(with $\lambda, \mu, \nu = 1, 2$). These "dotted" indices are still in use today. Van der Waerden also showed in analogy to the classical tensor calculus how the indices are pulled up and down:

$$a^1 = a_2, a^2 = -a_1 \quad \text{and} \quad a^{\dot{1}} = a_{\dot{2}}, a^{\dot{2}} = -a_{\dot{1}}.$$

Today, this relation is denoted typically with the help of the skew-symmetric $\epsilon-$tensor

$$(\epsilon^{\lambda\mu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as $a^\lambda = \epsilon^{\lambda\mu} a_\mu$.[17] So, van der Waerden outlined the foundations of the spinor calculus.

Then van der Waerden established a $2:1$ homomorphism between $SL_2\mathbb{C}$ (considered as a real Lie group) and the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ by giving a concrete mapping between spinors of the form $a_{\dot{\lambda}\mu}$ and the coordinates of a real four-dimensional vector space. In other words, he gave a bijection between 2×2 Hermitian matrices and Minkowski space. Thus, one got an

[16] "Die Aufgabe, alle ‚Größen' zu finden, die bei Lorentztransformationen nach irgendeiner Regel linear mit-transformiert werden, so daß bei Zusammensetzung zweiter [!] Lorentztransformationen auch die zugehörigen Transformationen der ‚Größen' zusammengesetzt werden, d.h. so daß dem Produkt zweier Lorentztransformationen wieder das Produkt entspricht, ist nichts anderes als das Problem der *Darstellung* der Lorentzgruppe durch lineare Transformationen."

[17] Van der Waerden introduced the $\epsilon-$tensor as a "pure spinor" into spinor calculus to construct invariants [van der Waerden, 1932, p.86f].

irreducible representation of $SL_2\mathbb{C}$ and a "two-valued" (zweideutige) representation of $\mathcal{L}_+^\uparrow$ where one could assign $\pm A \in SL_2\mathbb{C}$ to one Lorentz-transformation. For his construction van der Waerden could rely on the work of Weyl on representation theory and of E. A. Weiß on invariant theory.[18] Thus, mathematically, it was not a great challenge.

Van der Waerden then applied this calculus to physics: he translated the relativistic Dirac wave equation into spinor formalism. He started out with a slightly modified version of this wave equation given by Weyl in his book[19]

$$\frac{1}{c}\left(\frac{h}{i}\frac{\partial}{\partial t} + \Phi_0\right)\psi + \sum_{r=1}^{3} s_r'\left(\frac{h}{i}\frac{\partial}{\partial x_r} + \Phi_r\right)\psi + mc\Gamma_0\psi = 0,$$

where the wave function ψ has four components $\psi_i : \mathbb{R}^4 \longrightarrow \mathbb{C}$. Translating this equation step-by step into spinor calculus he ended up with a pair of spinor equations corresponding to the first two and the last two rows of Weyl's equation:

$$\begin{aligned} -\left(\tfrac{h}{i}\partial^{\dot\lambda}_{\ \mu} + \Phi^{\dot\lambda}_{\ \mu}\right)\psi_{\dot\lambda} + mc\chi_\mu &= 0 \\ \left(\tfrac{h}{i}\partial_{\dot\mu}^{\ \lambda} + \Phi_{\dot\mu}^{\ \lambda}\right)\chi_\lambda + mc\psi_{\dot\mu} &= 0, \end{aligned}$$

where the wave function consists of four spinor-components $\psi = (\psi_{\dot 1}, \psi_{\dot 2}, \chi_1, \chi_2)$ that are complex-valued functions and correspond to the irreducible representations of $SL_2\mathbb{C}$, and Φ is a field.

Van der Waerden did not stop there, but outlined the general spinor form of wave equations of the first order with a two-component wave-function, and of the second order with a two-component and a four-component wave function. Thus, he provided the physicists with a multitude of different forms of wave equations in spinor formalism. By doing so, he was able to answer Ehrenfest's question why a two-component wave function together with a wave equation of the first order would not suffice to describe the electron relativistically. Van der Waerden showed that a wave equation with a two-component wave-function of the first order

$$\partial_{\dot\lambda\mu}\psi^\mu + c_{\dot\lambda\mu}\psi^\mu = 0,$$

[18] Weiß [1924]; Weyl [1925, 1926, 1931].

[19] van der Waerden [1929, p.106f], Weyl [1928, p.172]. The constant h is Planck's constant divided by 2π (today's notation is $\hbar$), m is the mass of the electron, c the velocity of light, Φ_k denotes the electrostatic potential and the four-by-four matrices s_r', are built up by Pauli spin matrices $s_r (r = 1, 2, 3)$ as follows:

$$s_r' = \begin{pmatrix} s_r & 0 \\ 0 & -s_r \end{pmatrix}$$

with

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The four-by-four matrix Γ_0 is of the form

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}.$$

The matrices s_r' correspond up to a change of bases to Dirac's γ_k-matrices.

would imply that the mass of the electron is zero, which rules out this possibility.[20] Later, Ehrenfest used this very question in Casimir's examination.[21] So, Ehrenfest seems to have been convinced by van der Waerden of the usefulness of spinor calculus. Later, in spring 1933, van der Waerden helped Infeld, a Polish physicist visiting Leipzig, to introduce a spinor formalism for general relativity theory in order to find an alternative to the "$n-$Bein"-formalism which Leipzig physicists preferred no to use.[22] This shows van der Waerden as a very pragmatic scientist who respected the wishes of his colleagues and who made an effort to adapt to their special needs.

How was the spinor calculus received by Ehrenfest and other physicists? Uhlenbeck, a former student of Ehrenfest, together with O. Laporte published an article in the *Physical Reviews* - the American journal in which Slater had had his group-free method published two years before - advocating the use of spinor calculus.[23] They stated the rules of the calculus very clearly and applied it to the Dirac equation as well as to the Maxwell equations. Ehrenfest, however, was still not fully satisfied. What he wanted was an easy introduction to both tensors and spinors, and he also had some more conceptual questions.[24] It was he who urged A. Einstein and W. Mayer to develop their alternative concept of semivectors.[25] Later, in 1936, the spinor calculus was used by Dirac to derive wave equations for particles with spin greater than one half, that means wave equations for (elementary) particles that had not been discovered up to then.[26] And it is probably this publication by this prominent researcher which made the spinor calculus known to a wider audience and ensured that it was not forgotten.

Modern representation theory: Groups with operators

The next example will show that van der Waerden did not refrain from using modern algebraic concepts and methods in his group-theoretic monograph on quantum mechanics.[27] His introduction to representation theory was based on the concept "group with operators", a concept that had not appeared in any of the other articles or books on group-theoretic methods in quantum mechanics before. So, van der Waerden's approach was unique in this respect.

[20] The correspondence between Ehrenfest and van der Waerden shows that van der Waerden started out with a different equation. The published equation corresponds to Weyl's equation [Weyl, 1929, p.351] which van der Waerden may have come across in Göttingen in summer 1929. It was later used by Pauli to describe the neutrino.

[21] Ehrenfest to Uhlenbeck, 1.6.1930 [MB, ESC 10, S.2, 78].

[22] Van der Waerden to Schouten, 6.6.1933 [Centrum voor Wiskunde en Informatica (Amsterdam), correspondence Schouten. I thank Gerard Alberts for drawing my attention to this archive.] On the history of unified field theory see Goenner [2004].

[23] Laporte and Uhlenbeck [1931].

[24] Ehrenfest [1932].

[25] van Dongen [2004].

[26] Dirac [1936]. The elementary particles electron, positron, neutron, proton, known at that time, all had spin one half. However, from 1928 onwards it was known that the nitrogen nucleus had spin one by precision measurements performed by R. de Laer Kronig in Utrecht.

[27] van der Waerden [1932, chap. 2].

The concept "group with operators" was first introduced in 1925 by W. Krull.[28] Krull had used a different name for it: generalized finite abelian group (verallgemeinerte endliche Abelsche Gruppe). O. Schmidt and Noether took up the concept and made it more general.[29] It was Noether who introduced the name "group with operators" in a very general setting. Van der Waerden also used the concept in his book on modern algebra.[30]

In his introduction to group theory in his monograph on quantum mechanics van der Waerden made use of the concept group with operators only in the restricted sense introduced by Krull. His definition went as follows. Van der Waerden firstly defined a group. The definition he gave there was axiomatic and general and thus in line with modern algebra[31]:

> A set $\mathfrak{g}$ of *elements* $a, b, \ldots$ of any kind (e.g. of numbers, of linear transformations) is called a *group* if the following four conditions are satisfied:
>
> (8.1.) A "product" $a \cdot b$ (or ab) is assigned to each pair of elements a, b in such a way that it belongs again to $\mathfrak{g}$.
>
> (8.2.) The law of associativity $ab \cdot c = a \cdot bc$.
>
> (8.3.) There exists a "unit element", e or 1, with the property $ae = ea = a$.
>
> (8.4.) For each a of $\mathfrak{g}$ there exists an inverse a^{-1} in $\mathfrak{g}$, so that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ holds.
>
> The group is called *abelian*, if $ab = ba$ always applies.
>
> [...] Generally one speaks of a *group with operators* if certain "multipliers" or "operators" θ with the property (8.5) [i.e.
>
> $$\theta(u + v) = \theta u + \theta v \qquad (u, v \in \mathfrak{g})]$$
>
> are added to an abelian group.[32]
> [van der Waerden, 1932, p.28f, emphasis in the original]

Notice that there are very few restrictions on the set of operators.

Van der Waerden applied the concept of group with operators to the representation space. The representation space is a vector space, thus its vectors form

[28] Krull [1925, 1926].

[29] Noether [1927]; Schmidt [1928]; Noether [1929].

[30] van der Waerden [1930, 1931].

[31] On the development of modern algebra see e. g. Corry [1996].

[32] „Eine Menge $\mathfrak{g}$ von *Elementen* $a, b, \ldots$ irgendwelcher Art (z.B. von Zahlen, von linearen Transformationen) heißt eine *Gruppe*, wenn folgende vier Bedingungen erfüllt sind:

(8.1.) Jedem Elementenpaar a, b ist ein „Produkt" $a \cdot b$ (oder ab), das wieder zu $\mathfrak{g}$ gehört, zugeordnet.

(8.2.) Das Assoziativgesetz: $ab \cdot c = a \cdot bc$.

(8.3.) Es gibt ein „Einselement", e oder 1, mit der Eigenschaft $ae = ea = a$.

(8.4.) Zu jedem a von $\mathfrak{g}$ existiert ein Inverses a^{-1} in $\mathfrak{g}$, so daß $a \cdot a^{-1} = a^{-1} \cdot a = 1$ ist.

Die Gruppe heißt *Abelsch*, wenn stets $ab = ba$ ist. [...] Man redet im allgemeinen, wenn zu einer additiven [d. i. Abelschen] Gruppe gewisse „Multiplikatoren" oder „Operatoren" θ mit der Eigenschaft (8.5.) [d. i. $\theta(u + v) = \theta u + \theta v$] hinzugenommen werden, von einer *Gruppe mit Operatoren.*"

an abelian group with respect to addition. The set of operators are the scalars together with the representation matrices of the group in question. This makes the representation space into a group with operators. Van der Waerden showed how the central concepts of representation theory, such as invariant subspaces or irreducible representation, can be deduced from the theory of groups with operators.

One advantage of groups with operators was that van der Waerden could easily prove an important uniqueness theorem: If a representation splits into irreducible representations then this splitting is unique up to isomorphism. This was a central theorem of representation theory. It was also vital for quantum mechanics because of the correspondence between irreducible representations and quantum numbers. Proving the uniqueness theorem was elementary in this general setting.[33]

Wigner and Weyl also mentioned this central theorem of group theory. Wigner maintained it in full generality, but only proved it for groups with a finite number of elements.[34] He used so-called transcendental methods relying on characters. This approach went back to G. Frobenius and I. Schur.[35] It had the advantage of achieving a constructive method for reducing a given representation into irreducibles. This was also how Weyl had proceeded in the first edition of his book. However, in the second edition he changed to a more modern approach.[36] Although he did not introduce the concept of group with operators he used it implicitly. Weyl thought that this approach was more elementary and that it allowed a "full insight" (vollen Einblick) into the situation and a "complete understanding of the context" (restloses Verständnis der Zusammenhänge).[37]

Van der Waerden also saw another advantage of the concept of groups with operators, especially in physical contexts. Groups with operators allowed an easy notation of the group operation. Instead of working with different symbols to denote different representations of the same group, one could simply use the group element to operate on the representation space. Thus the notation becomes somewhat simplified.

The concept of group with operators allowed an easy and short introduction to representation theory from the point of view of modern algebra. This kind of approach was quite original. However, as far as I know, it was not followed up by anyone else. Van der Waerden used it in later editions of the book, even though the concept had come out of fashion. Van der Waerden's approach was not fully modern, since he did not develop the theory in its full generality, but only in so far as it was necessary to introduce representation theory of groups. Thus, he tailored his modern approach to the needs of physicists.

[33] van der Waerden [1932, §11].
[34] Wigner [1931, p.95].
[35] On the history of the representation theory of Lie groups see Hawkins [2000].
[36] Weyl [1931, chap. III, §6].
[37] Weyl [1931, p.VIf.].

Slater's method revisited

In the letter to van der Waerden mentioned earlier, Ehrenfest characterized the appearance of group theory in quantum mechanics as "group plague" (Gruppenpest). As already mentioned he did not intend to get rid of it altogether - as one would with a real outbreak of the plague, but he tried to master this new set of mathematical tools. The resistance to group theory grew when the young American physicist J. C. Slater introduced a method in 1929 to determine the multiplet system of an atom with several electrons without using group theory.[38] Slater's method was warmly welcomed by the physicists' community. Many physicists believed that this was the beginning of the end of group theory in quantum mechanics. Slater remembered this as follows:

> As soon as this paper [Slater, 1929] became known, it was obvious that a great many other physicists were disgusted as I [Slater] had been with the group-theoretical approach to the problem. As I heard later, there were remarks made such as "Slater has slain the '*Gruppenpest.*' " I believe that no other piece of work I had done was so universally popular. [Slater, 1975, p.62]

Slater's method rested on an approach developed by Hund who later became a colleague of van der Waerden in Leipzig.[39]

Van der Waerden introduced two methods to determine the multiplet structure of an atom with several electrons in his text book. He gave a very concise summary of the group-theoretical procedure without going into details but referring the reader instead to Weyl's text book. Van der Waerden then turned to Slater's alternative method. He introduced it in a positive light:

> However, there is a second method, in principle already older, recently applied successfully in particular by J.C. Slater, which gets by with much simpler aids and which does not require the representation theory of the permutation group.[40] [van der Waerden, 1932, p.120, emphasis in the original]

Van der Waerden went on to explain Slater's method.[41] Group-theoretically speaking, Slater could avoid the permutation group by transforming the spin and the "place" of the electrons simultaneously and by taking only those configurations into account which give rise to antisymmetric wave functions. In a group-theoretic approach the permutation group could operate on both spaces separately. Then van der Waerden described Slater's method, which was based on a table of configurations, then converted into a graphical diagram and graphically analyzed. Instead of just copying this method, van der Waerden optimized it. His main achievement was to do away with the graphical part and to develop

[38] Slater [1929].
[39] Hund [1927].
[40] "Es gibt aber eine zweite, im Prinzip schon ältere, neuerdings vor allem von J.C. Slater erfolgreich angewendete Methode, die mit viel einfacheren Hilfsmitteln auskommt und insbesondere die Darstellungstheorie der Permutationsgruppe nicht benötigt."
[41] van der Waerden [1932, p.120-124].

a purely computational algorithm instead. Firstly, van der Waerden shortened the table of configurations so that it only contained configurations that gave rise to an antisymmetric wave function and so that all configurations with negative values of $M_S = \sum m_s$ were left out due to reasons of symmetry.

The following table (Table 1) shows van der Waerden's table of configuration for three electrons with $n = 2$ and $l = 1$, i.e. three $2p$ electrons.[42] The content

			M_L	M_S
(2 1 1 +)	(2 1 0 +)	(2 1 -1 +)	0	$\frac{3}{2}$
(2 1 1 +)	(2 1 0 +)	(2 1 1 -)	2	$\frac{1}{2}$
(2 1 1 +)	(2 1 0 +)	(2 1 0 -)	1	$\frac{1}{2}$
(2 1 1 +)	(2 1 0 +)	(2 1 -1 -)	0	$\frac{1}{2}$
(2 1 1 +)	(2 1 -1 +)	(2 1 1 -)	1	$\frac{1}{2}$
(2 1 1 +)	(2 1 -1 +)	(2 1 0 -)	0	$\frac{1}{2}$
(2 1 1 +)	(2 1 -1 +)	(2 1 -1 -)	-1	$\frac{1}{2}$
(2 1 0 +)	(2 1 -1 +)	(2 1 1 -)	0	$\frac{1}{2}$
(2 1 0 +)	(2 1 -1 +)	(2 1 0 -)	-1	$\frac{1}{2}$
(2 1 0 +)	(2 1 -1 +)	(2 1 -1 -)	-2	$\frac{1}{2}$

Table 1: Van der Waerden's table of configurations for three $2p-$electrons

of the round brackets symbolizes an electron with quantum numbers n, l, m_l and spin $\pm 1/2$. The left-hand side of each row is an abbreviation for an antisymmetric wave function which is given by the following expression (e.g. for the first row):

$$\sum_{P \in S_3} \delta_P P \psi(2\ 1\ 1\ |q_1)\psi(2\ 1\ 0\ |q_2)\psi(2\ 1\ -1\ |q_3) u_1 v_1 w_1$$

in which S_3 is the permutation group of three elements, δ_P is the sign of the permutation, q_f a system of space coordinates of the $f-$th electron ($f = 1, 2, 3$) and $u_i, v_j, w_k (i, j, k = 1, 2)$ are vector components in spin space with a 1 in the index indicating spin $+1/2$ and a 2 indicating spin $-1/2$. In the second column one adds up the m_l and in the third the spin of the electron configuration, giving rise to numbers M_L and M_S.

Secondly, van der Waerden gave a clear-cut procedure of how to find the pairs L, S of an arising multiplet directly from the table of configurations. The procedure was as follows: First, one has to choose the greatest value for M_S in the table (in this case 3/2). Together with the value M_L from the same row (in this case 0) it gives rise to a multiplet with $L = M_L$ and $S = M_S$. In case of more rows with the same value M_S, you choose the row with the greatest value M_L. Then you delete from the table all rows with values $M_S = S, S-1, \ldots, 0$ and $M_L = L, L-1, \ldots, -L$ once. (In the example of $S = 3/2, L = 0$, one deletes two rows: 1st, 4th). Then you start the procedure again from the beginning. In the example this leads to the multiplets ${}^4S,\ {}^2D,\ {}^2P$.

[42] van der Waerden [1932, p.121].

Van der Waerden's third way of optimizing Slater's method was to give three rules. Some of these rules were already implicitly used by Slater in his examples. The first rule stated that one could neglect the electron of a full shell (same n, l for $2(2l + 1)$ electrons). The second one described how to construct the multiplets of the whole system from multiplets arising from subsystems. As subsystems he considered electrons in the same shell, i.e. sets of equivalent electrons (same n, l). The third rule made use of the symmetry of the multiplet structure within a shell to determine the multiplets. At the end of the passage, van der Waerden gave a list of the possible multiplets occurring in shells up to $l = 2$. This list could be used as a table of reference by the working physicist when determining the multiplet structure of an atom with several electrons.

So, van der Waerden took on board the concerns of those physicists who were interested in easy calculational techniques. He restricted the use of group theory to a minimum. When it was possible to use a mathematically simpler method he did not hesitate to do so. This again is a very pragmatic attitude, which is not at all in line with the modern algebraic way of reasoning. The inclusion of Slater's method into his book might also indicate that it was a method used by the physicists in Leipzig - after all Slater's method was based on method developed by Hund.

Van der Waerden's approach to Slater's method also differed considerably from Weyl's and Wigner's. Both of them explained the group-theoretic approach and gave all the mathematical details. Weyl did not mention Slater's method explicitly in his second edition of 1931, but implicitly referred to it.[43] In his preface Weyl alluded also to Slater's method:

> It has recently been said that the "group plague" will gradually be taken out of quantum physics. This is definitely not true with respect to the group of rotations and of Lorentz-transformations. As to the group of permutations, its study really seems to include a detour due to the Pauli principle. Nevertheless, the representations of the group of permutations must remain a natural tool of theory, as long as one takes the existence of spin into account, but neglects its dynamic effect and as long as one wants to have a general overview of the resulting circumstances.[44] [Weyl, 1931, p.viif]

[43] Slater's article was mentioned in two footnotes but only because of Slater's innovative approach in perturbation theory to calculate energy levels [Weyl, 1931, p.173, fn. 4; p.314, fn. 15].

[44] "Es geht in jüngster Zeit die Rede, daß die „Gruppenpest" allmählich wieder aus der Quantenphysik ausgeschieden wird. Dies ist gewiss unrichtig, bezüglich der Rotations- und Lorentz-Gruppe. Was die Permutationsgruppe anlangt, so scheint ihr Studium in der Tat wegen des Pauliverbots einen Umweg einzuschließen. Dennoch müssen die Darstellungen der Permutationsgruppe ein natürliches Werkzeug der Theorie bleiben, solange die Existenz des spins berücksichtigt, seine dynamische Einwirkung aber vernachlässigt wird und man die daraus resultierenden Verhältnisse allgemein überblicken will." The given translation is closer to the German original than that of the English edition of 1950. The appreciative mentioning of Slater's works together with those of D. R. Hartree and Dirac by Weyl at the end of the paragraph (from which the above quote was taken) is done in the context of *numerical* methods in perturbation theory and thus, in my opinion, does not refer to Slater's method to avoid group-theoretic reasoning (see also previous footnote).

Weyl acknowledged that the permutation group could be avoided - at least to a certain extent - because of the Pauli principle, a principle which was also at work in Slater's method. Yet, he also insisted that the representation theory of the permutation group could not be avoided generally. Van der Waerden presented Slater's method as a short-cut around the representation theory of the permutation group for determining the multiplet system.

What about Wigner? Like van der Waerden, he mentioned Slater's theory. However, Wigner explained why Slater's method worked from a group-theoretic point of view. He groupified Slater's method, so to speak. In this way, he could also determine its range of applicability. Slater's method was restricted to particles of spin one half. The group-theoretic method, however, worked for particles with arbitrary spin. So, Wigner pointed to a physical reason which spoke in favour of the more complicated group-theoretic method.

Conclusion

The examples chosen to illustrate van der Waerden's approach failed to give a really coherent picture. The first example of the spinor calculus shows that van der Waerden developed a calculus at the request of Ehrenfest, that he tried to use as little mathematics as possible. No modern algebra went into this work. Instead a calculus was modelled on the tensor calculus. The second example - groups with operators - shows that he confronted the physicists with a new concept of modern algebra. However, the purpose of the concept was only to give an introduction to representation theory. Moreover, van der Waerden used the concept in a very limited way, adapted to the representation theory of groups. The third example shows that van der Waerden was a pragmatic person who included and improved Slater's method for reasons of simplicity. This is quite remarkable for a book dealing with group-theoretic methods.

What can we conclude from this? Van der Waerden did not aim to apply modern algebraic methods to quantum mechanics. It was not his main interest to demonstrate the power of group theory. I think he really intended to help those physicists like Ehrenfest to understand these new methods and to be able to work with them. Of course, by doing so he helped to spread and advance the new method.

However, I think, this was not intended as "fighting back" as J. Mehra and H. Rechenberg put it pointedly.[45] In my opinion, it was intended as a contribution to assist physicists - not in an arrogant way, but rather in a friendly, helpful way. This is indicated by the influence of "local" physicists on van der Waerden's research.[46]

[45] Mehra and Rechenberg [chap. III.4(e) 2000] .

[46] Another example of this influence is the chapter on molecular spectra in [van der Waerden, 1932] which apparently was influenced by discussions with Hund [Kleint and Wiemers, 1993, p.205, fn. 26].

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