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# Pascual Jordan's resolution of the conundrum of the wave-particle duality of light

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In 1909, Einstein derived a formula for the mean square energy fluctuation in a small subvolume of a box filled with black-body radiation. This formula is the sum of a wave term and a particle term. Einstein concluded that a satisfactory theory of light would thus have to combine aspects of a wave theory and a particle theory. In the following decade, various attempts were made to recover this formula without Einstein's light quanta. In a key contribution to the 1925 *Dreimännerarbeit* with Born and Heisenberg, Jordan showed that a straightforward application of Heisenberg's *Umdeutung* procedure, which forms the core of the new matrix mechanics, to a system of waves reproduces both terms of Einstein's fluctuation formula. Jordan thus showed that these two terms do not require separate mechanisms, one involving particles and one involving waves. In matrix mechanics, both terms arise from a single consistent dynamical framework.

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## Outline

**Einstein, fluctuations, light quanta, and wave-particle duality**

**Resistance to the light-quantum hypothesis**

**Jordan’s resolution of the wave-particle conundrum in 3M**

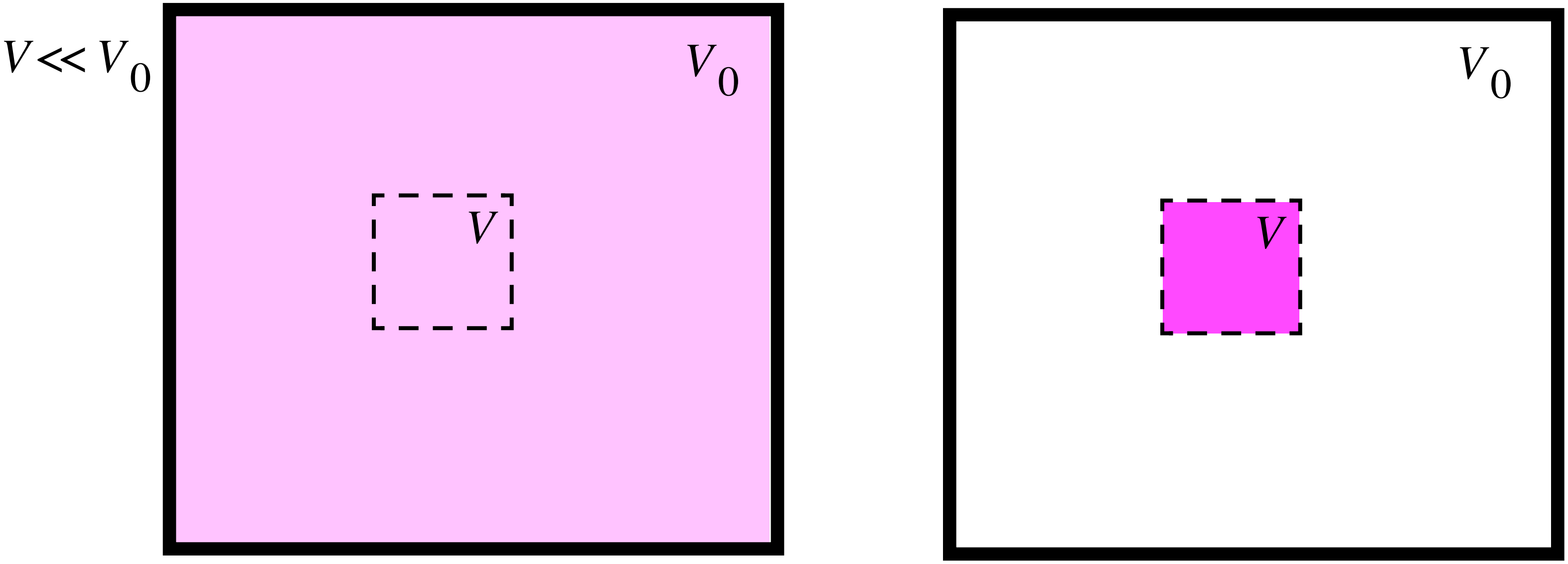
**Jordan’s derivation of Einstein’s fluctuation formula: the gory details**

**Jordan on the importance of his result**

**Conclusions**

# Einstein, fluctuations, light quanta, and wave-particle duality

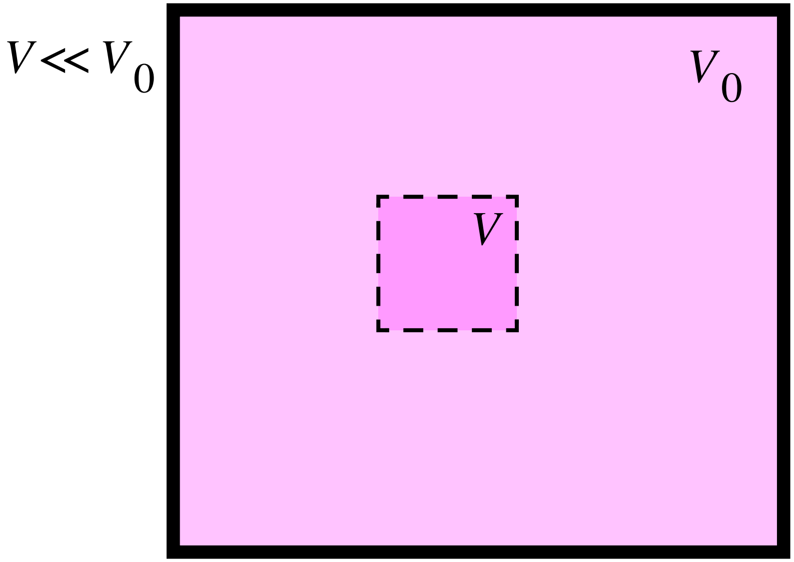
**1905: Einstein’s ‘first fluctuation formula’** (Jordan’s terminology) → **light quanta**



Entropy of black-body radiation in the Wien regime (high frequency) and Boltzmann’s principle $S = k \log W$:

- Probability of finding all black-body radiation in $V_0$ in subvolume $V$: $(V/V_0)^{E/h\nu}$
- Probability of finding all $N$ molecules of ideal gas in $V_0$ in subvolume $V$: $(V/V_0)^N$
- Einstein’s conclusion (“the miraculous argument” [Norton]): “radiation ... behaves ... as if it consisted of [$N$] mutually independent energy quanta of magnitude [$h\nu$].”

# 1909: Einstein's 'second fluctuation formula' → wave-particle duality



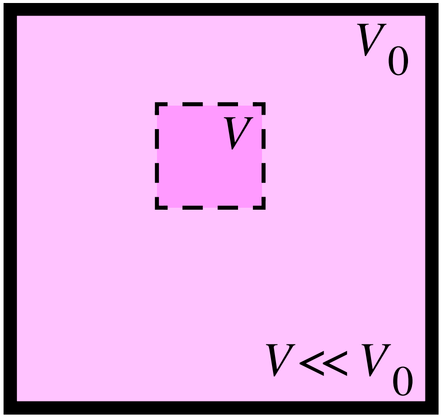
**1904.** Einstein applies formula for mean square energy fluctuation (Boltzmann, Gibbs):

$$\langle \Delta E^2 \rangle = kT^2 \frac{d\langle E \rangle}{dT}$$

to black-body radiation (final installment of the 'statistical trilogy'; cf. Ryno-Renn, Norton, Uffink in *Studies in History and Philosophy of Modern Physics* 37 (2006) 1 [Centenary of Einstein's *Annus Mirabilis*])

**1909.** Einstein's *GDNA* lecture in Salzburg. Martin J. Klein:* "Instead of trying to derive the distribution law from some more fundamental starting point, he turned the argument around. Planck's law had the solid backing of experiment; why not assume its correctness and see what conclusions it implied as to the structure of radiation?"

* "Einstein and the Wave-Particle Duality." *The Natural Philosopher* 3 (1964): 1–49.



**Task:** find the mean square fluctuation in subvolume $V$ of the energy of black-body radiation at temperature $T$ with spectral distribution $\rho(\nu, T)$ in frequency range $(\nu, \nu + \Delta\nu)$.

Introduce: $\langle E_\nu \rangle = \rho(\nu, T) V \Delta\nu$, the average energy in frequency range $(\nu, \nu + \Delta\nu)$ in subvolume $V$.

Frequency-specific fluctuation formula: $\langle \Delta E_\nu^2 \rangle = kT^2 \dfrac{\partial \rho(\nu, T)}{\partial T} V \Delta\nu$.

Results for different black-body radiation laws:

| | | | |
|---|---|---|---|
| Rayleigh-Jeans | $\rho^{\mathrm{RJ}}(\nu, T) = \dfrac{8\pi}{c^3} \nu^2 kT$ | $\langle \Delta E_\nu^2 \rangle_{\mathrm{RJ}} = \dfrac{c^3}{8\pi\nu^2} \dfrac{\langle E_\nu \rangle_{\mathrm{RJ}}^2}{V\Delta\nu}$ | **waves** |
| Wien | $\rho^{\mathrm{W}}(\nu, T) = \dfrac{8\pi h}{c^3} \nu^3 e^{-h\nu/kT}$ | $\langle \Delta E_\nu^2 \rangle_{\mathrm{W}} = h\nu \langle E_\nu \rangle_{\mathrm{W}}$ | **particles** |
| Planck | $\rho^{\mathrm{P}}(\nu, T) = \dfrac{8\pi h}{c^3} \dfrac{\nu^3}{e^{h\nu/kT} - 1}$ | $\langle \Delta E_\nu^2 \rangle_{\mathrm{P}} = \dfrac{c^3}{8\pi\nu^2} \dfrac{\langle E_\nu \rangle_{\mathrm{P}}^2}{V\Delta\nu} + h\nu \langle E_\nu \rangle_{\mathrm{P}}$ | **waves**<br>**+**<br>**particles!** |

Einstein's conclusion (1909): "the next phase of the development of theoretical physics will bring us a theory of light that can be interpreted as a kind of fusion of the wave and emission theories."

## Resistance to the light-quantum hypothesis

- Planck and Nernst when recruiting Einstein for the Prussian Academy in Berlin (1914): "That he may sometimes have missed the target in his speculations, as for example, in his hypothesis of light quanta, cannot really be held against him."

- Millikan after verifying Einstein's predictions for the photoelectric effect (1916): "We are confronted by the astonishing situation that these facts were correctly and exactly predicted nine years ago by a form of quantum theory which has now been pretty generally abandoned." (cf. Roger Stuewer in *The Cambridge Companion to Einstein* [in preparation])

- Mie to Wien, 6 February 1916: "It is as if his temperamental wishful thinking occasionally robs him of his capacity for calm deliberation, as once with his new theory of light, when he believed he could construct a light beam capable of interference out of nothing but incoherent parts."

# Resistance to the light-quantum hypothesis persists even after the Compton effect.*

- Schizophrenic attitude of many physicists after Compton effect (1923): accept light quanta but continue to treat light as if it were a classical wave (particularly clear in dispersion theory: Ladenburg, Reiche, Van Vleck, Born, Heisenberg)

- Bohr's last stand against light quanta: BKS—the Bohr-Kramers-Slater theory (1924). BKS "explains" Compton effect without light quanta.

- After Bothe & Geiger and Compton & Simon decisively refute BKS in 1925, Slater's original idea of light quanta guided by a classical field is resurrected (similar ideas: De Broglie, Einstein, Swann).

  Slater (in letter to *Nature*, July 25, 1925): "The simplest solution to the radiation problem then seems to be to return to the view of a virtual field to guide corpuscular quanta."

  Kramers to Urey, July 16, 1925: "We [here in Copenhagen] think that Slater's original hypothesis contains a good deal of truth."

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*Duncan & Janssen. "On the Verge of *Umdeutung* in Minnesota: Van Vleck and the Correspondence Principle." *Archive for History of Exact Sciences* (forthcoming).

## Attempts to recover Einstein’s fluctuation formula without light quanta, 1909–1925

Albert Einstein and Ludwig Hopf, “Statistische Untersuchung der Bewegung eines Resonators in einem Strahlungsfeld.” *AdP* (= *Annalen der Physik*) 33 (1910): 1105–1115.

Max von Laue, “Ein Satz der Wahrscheinlichkeitsrechnung und seine Anwendung auf die Strahlungstheorie.” *AdP* 47 (1915): 853–878.

Albert Einstein, “Antwort auf eine Abhandlung M. v. Laues …” *AdP* 47 (1915): 879–885.

Max von Laue, “Zur Statistik der Fourierkoeffizienten der natürlichen Strahlung.” *AdP* 48 (1915): 668–680.

Max Planck, “Über die Natur der Wärmestrahlung.” *AdP* 73 (1924): 272–288.

Paul Ehrenfest, “Energieschwankungen im Strahlungsfeld oder Kristallgitter bei Superposition quantisierter Eigenschwingungen.” *ZfP* (= *Zeitschrift für Physik)* 34 (1925): 362–373.

Pascual Jordan, “Zur Theorie der Quantenstrahlung.” *ZfP* 30: (1924) 297–319.

Albert Einstein, “Bemerkung zu P. Jordans Abhandlung …” *ZfP* 31 (1925): 784–785.

Pascual Jordan, “Über das thermische Gleichgewicht zwischen Quantenatomen und Hohlraumstrahlung.” *ZfP* 33 (1925): 649–655

## Jordan's resolution of the wave-particle conundrum in 3M (= *Dreimännerarbeit*)

Max Born, Werner Heisenberg, Pascual Jordan, "Zur Quantenmechanik II." *Zeitschrift für Physik* 35 (1925): 557–615. English translation in B. L. Van der Waerden (ed.), *Sources of Quantum Mechanics*. New York: Dover, 1968.

**Einstein's fluctuation formula emerges from straightforward application of Heisenberg's *Umdeutung* procedure to classical waves. The formula does not require separate mechanisms, one involving particles and one involving waves. In matrix mechanics, both terms arise from a single consistent dynamical framework.**

Presented in 3M but actually Jordan's work:

Heisenberg to Pauli, October 23, 1925: "Jordan claims that the interference calculations come out right, both the classical and the Einsteinian terms … I am rather sorry that I do not understand enough statistics to be able to judge how much sense it makes; but I cannot criticize either, because the problem itself and the subsequent calculations sound meaningful."

Jordan to Van der Waerden, April 10, 1962: "[my] most important contribution to quantum mechanics."

# Other works by Jordan on fluctuations in radiation and on field quantization*

**Pascual Jordan (1902–1980)**

1924. “Zur Theorie der Quantenstrahlung.” *ZfP* 30: 297–319. Einstein’s response: “Bemerkung zu P. Jordans Abhandlung …” *ZfP* 31: 784–785.

1925. “Über das thermische Gleichgewicht zwischen Quantenatomen und Hohlraumstrahlung.” *ZfP* 33: 649–655

1927. “Über Wellen und Korpuskeln in der Quantenmechanik.” *ZfP* 45: 766–775.

1927 (with Oskar Klein), “Zum Mehrkörperproblem der Quantentheorie.” *ZfP* 45: 751–765.

1928 (with Eugene Wigner), “Über das Paulische Äquivalenzverbot.” *ZfP* 47: 631–651.

1928. “Die Lichtquantenhypothese. Entwicklung und gegenwärtiger Stand.” *Ergebnisse der exakten Naturwissenschaften* 7: 158–208.

1930 (with Max Born), *Elementare Quantenmechanik*. Berlin: Springer.

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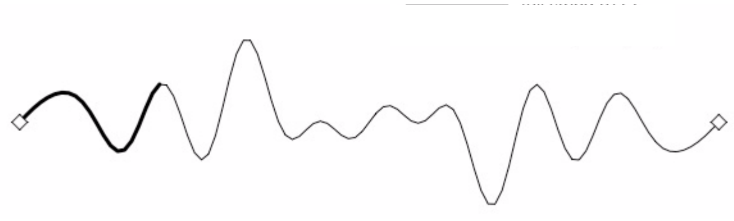
*Cf. Olivier Darrigol, “The origin of quantized matter waves.” *Historical Studies in the Physical and Biological Sciences* 16 (1986): 197–253.

# Jordan’s derivation of Einstein’s fluctuation formula: the gory details

**Classical model (Ehrenfest): String of length *l* fixed at both ends (= 1D analogue of EM field in box with conducting sides).** Displacement $u(x, t)$

Hamiltonian: $H = \frac{1}{2}\int_0^l dx(\dot{u}^2 + u_x^2)$



Insert Fourier expansion $u(x, t) = \sum_{k=1}^{\infty} q_k(t)\sin(\omega_k x)$ with Fourier coefficients $q_k(t) = a_k\cos(\omega_k t + \varphi_k)$ and frequencies $\omega_k \equiv k\left(\frac{\pi}{l}\right)$ :

$$H = \frac{1}{2}\int_0^l dx \sum_{j,k=1}^{\infty} (\dot{q}_j(t)\dot{q}_k(t)\sin(\omega_j x)\sin(\omega_k x) + \omega_j\omega_k q_j(t)q_k(t)\cos(\omega_j x)\cos(\omega_k x))$$

Only contributions for $j = k$: $\{\sin(\omega_j x)\}$ orthogonal on $(0, l)$: $\int_0^l dx\sin\omega_j x\sin\omega_k x = \frac{l}{2}\delta_{jk}$

*H* turns into Hamiltonian for infinite set of uncoupled oscillators of mass *l*/2:

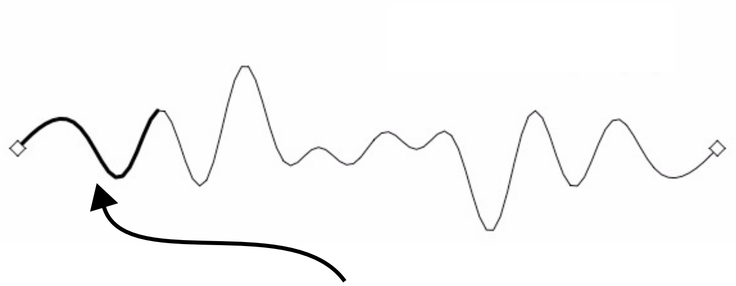
$$H = \sum_{j=1}^{\infty} \frac{l}{4}(\dot{q}_j^2(t) + \omega_j^2 q_j^2(t)) = \sum_{j=1}^{\infty} H_j$$

Hamiltonian for string turns into Hamiltonian for infinite set of uncoupled oscillators.

$$H = \frac{1}{2}\int_0^l dx(\dot{u}^2 + u_x^2) = \sum_{j=1}^{\infty} H_j \text{ with } H_j = \frac{l}{4}(\dot{q}_j^2(t) + \omega_j^2 q_j^2(t))$$

- Distribution of energy over frequencies is constant in time.
- Distribution of energy in some frequency range over length of string varies in time.

**Task: calculate the mean square fluctuation of the energy in a narrow frequency range $(\omega, \omega + \Delta\omega)$ in a small segment $(0, a)$ of the string ($a \ll l$).**



small segment $(0, a)$ of the string

**Two steps**

1. **(Jordan’s version of) Ehrenfest’s classical calculation**

2. **Jordan’s quantum-mechanical version of the calculation**

# 1. Classical calculation of mean square fluctuation of energy in $(\omega, \omega + \Delta\omega)$ in $(0, a)$

## Instantaneous energy in $(\omega, \omega + \Delta\omega)$ in $(0, a)$

$$E_{(\omega, a)}(t) = \frac{1}{2}\int_0^a dx \sum_{\substack{\omega < j(\pi/l) < \omega + \Delta\omega \\ \omega < k(\pi/l) < \omega + \Delta\omega}} \{\dot{q}_j(t)\dot{q}_k(t)\sin(\omega_j x)\sin(\omega_k x) + \omega_j\omega_k q_j(t)q_k(t)\cos(\omega_j x)\cos(\omega_k x)\}$$

$\{\sin(\omega_j x)\}$ *not* orthogonal on $(0, a) \rightarrow$ Both $(j = k)$-terms and $(j \neq k)$-terms contribute.

- **$(j = k)$-terms: average energy in $(\omega, \omega + \Delta\omega)$ in $(0, a)$:**
  - $E^{(j=k)}_{(\omega, a)}(t) = \frac{a}{4}\sum_j (\dot{q}^2_j(t) + \omega^2_j q^2_j(t)) = \frac{a}{l}\sum_j H_j$
  - $H_j$'s are constant $\rightarrow E^{(j=k)}_{(\omega, a)}(t) = \overline{E^{(j=k)}_{(a, \omega)}(t)}$ (bar means time average)
  - For $j \neq k$, $\overline{\dot{q}_j(t)\dot{q}_k(t)} = \overline{q_j(t)q_k(t)} = 0 \rightarrow \overline{E^{(j\neq k)}_{(\omega, a)}(t)} = 0$

  $\rightarrow$ $\boldsymbol{E^{(j=k)}_{(\omega, a)}(t) = \overline{E_{(\omega, a)}(t)}}$ = fraction $(a/l)$ of the (constant) total amount of energy in this frequency range in the entire string

- **$(j \neq k)$-terms: instantaneous fluctuation of energy in $(\omega, \omega + \Delta\omega)$ in $(0, a)$:**

  $\rightarrow$ $\boldsymbol{E^{(j\neq k)}_{(\omega, a)}(t) = E_{(\omega, a)}(t) - \overline{E_{(\omega, a)}(t)} = \Delta E_{(\omega, a)}(t)}$

# Instantaneous energy fluctuation in $(\omega, \omega + \Delta\omega)$ in $(0, a)$

$$\Delta E_{(\omega, a)}(t) = E_{(\omega, a)}(t) - \overline{E_{(\omega, a)}(t)} = E^{(j \neq k)}_{(\omega, a)}(t)$$

$$= \int_0^a dx \sum_{j \neq k} (\dot{q}_j \dot{q}_k [\cos((\omega_j - \omega_k)x) - \cos((\omega_j + \omega_k)x)]$$

$$+ \omega_j \omega_k q_j q_k [\cos((\omega_j - \omega_k)x) + \cos((\omega_j + \omega_k)x)])$$

$$= \frac{1}{4} \sum_{j \neq k} \left( \dot{q}_j \dot{q}_k \left[ \overbrace{\frac{\sin((\omega_j - \omega_k)a)}{\omega_j - \omega_k} - \frac{\sin((\omega_j + \omega_k)a)}{\omega_j + \omega_k}}^{\equiv K_{jk}} \right] \right.$$

$$\left. + \omega_j \omega_k q_j q_k \left[ \overbrace{\frac{\sin((\omega_j - \omega_k)a)}{\omega_j - \omega_k} + \frac{\sin((\omega_j + \omega_k)a)}{\omega_j + \omega_k}}^{\equiv K'_{jk}} \right] \right)$$

**Result** (suppress time dependence):

$$\Delta E_{(\omega, a)} = \frac{1}{4} \sum_{j \neq k} (\dot{q}_j \dot{q}_k K_{jk} + \omega_j \omega_k q_j q_k K'_{jk})$$

## Energy fluctuation in range $(\omega, \omega + \Delta\omega)$ in segment $(0, a)$:

**Instantaneous fluctuation**

$$\Delta E_{(\omega,\, a)} = \frac{1}{4}\sum_{j \neq k}(\dot q_j \dot q_k K_{jk} + \omega_j \omega_k q_j q_k K'_{jk}) \equiv \Delta E_{1(\omega,\, a)} + \Delta E_{2(\omega,\, a)}$$

**Mean square fluctuation**

$$\overline{\Delta E^2_{(\omega,\, a)}} = \overline{\Delta E^2_{1(\omega,\, a)}} + \overline{\Delta E^2_{2(\omega,\, a)}} + \overline{\Delta E_{1(\omega,\, a)}\Delta E_{2(\omega,\, a)}} + \overline{\Delta E_{2(\omega,\, a)}\Delta E_{1(\omega,\, a)}}$$

- **Square terms**

$$\overline{\Delta E^2_{1(\omega,\, a)}} + \overline{\Delta E^2_{2(\omega,\, a)}} =$$

$$\frac{1}{16}\sum_{j \neq k}\sum_{j' \neq k'}(\overline{\dot q_j \dot q_k \dot q_{j'} \dot q_{k'}} K_{jk} K_{j'k'} + \overline{q_j q_k q_{j'} q_{k'}}\omega_j \omega_k \omega_{j'} \omega_{k'} K'_{jk} K'_{j'k'})$$

- **Cross terms**

$$\overline{\Delta E_{1(\omega,\, a)}\Delta E_{2(\omega,\, a)}} + \overline{\Delta E_{2(\omega,\, a)}\Delta E_{1(\omega,\, a)}} =$$

$$\frac{1}{16}\sum_{j \neq k}\sum_{j' \neq k'}(\overline{\dot q_j \dot q_k q_{j'} q_{k'}}\omega_{j'} \omega_{k'} K_{jk} K'_{j'k'} + \overline{q_j q_k \dot q_{j'} \dot q_{k'}}\omega_j \omega_k K'_{jk} K_{j'k'})$$

Classically: only square terms contribute to $\overline{\Delta E^2_{(\omega,\, a)}}$. Quantum-mechanically: square terms *and cross terms* contribute.

## Evaluation of time averages of products of $q$’s and $\dot{q}$’s in square terms of $\overline{\Delta E^2_{(\omega,\, a)}}$

$$\overline{\Delta E^2_{1\,(\omega,\, a)}} + \overline{\Delta E^2_{2\,(\omega,\, a)}} = \frac{1}{16}\sum_{j \neq k}\sum_{j' \neq k'}\left(\overline{\dot{q}_j\dot{q}_k\dot{q}_{j'}\dot{q}_{k'}}K_{jk}K_{j'k'} + \overline{q_jq_kq_{j'}q_{k'}}\omega_j\omega_k\omega_{j'}\omega_{k'}K'_{jk}K'_{j'k'}\right)$$

- Recall $q_k(t) = a_k\cos(\omega_k t + \varphi_k)$.
- Argument for time averages $\overline{q_k(t)}$, $\overline{\dot{q}_k(t)}$ also works for phase averages $[q_k(t)]$, $[\dot{q}_k(t)]$ (square brackets: average over $\varphi_k$)

$$\left.\begin{array}{l} \overline{q_jq_kq_{j'}q_{k'}} \propto \overline{\cos(\omega_j t)\cos(\omega_k t)\cos(\omega_{j'} t)\cos(\omega_{k'} t)} \\ \overline{\dot{q}_j\dot{q}_k\dot{q}_{j'}\dot{q}_{k'}} \propto \overline{\sin(\omega_j t)\sin(\omega_k t)\sin(\omega_{j'} t)\sin(\omega_{k'} t)} \end{array}\right\} = 0 \text{ unless } (j = j', k = k') \text{ or } (j = k', k = j')$$

Cross terms (2 cosines, 2 sines):

$$\overline{q_jq_k\dot{q}_{j'}\dot{q}_{k'}} \propto \overline{\cos(\omega_j t)\cos(\omega_k t)\sin(\omega_{j'} t)\sin(\omega_{k'} t)} = 0 \text{ for all index combinations}$$

$$\overline{\Delta E^2_{(\omega,\, a)}} = \overline{\Delta E^2_{1\,(\omega,\, \alpha)}} + \overline{\Delta E^2_{2\,(\omega,\, \alpha)}} = \frac{1}{8}\sum_{j \neq k}\left(\overline{\dot{q}^2_j}\,\overline{\dot{q}^2_k}K^2_{jk} + \overline{q^2_j}\,\overline{q^2_k}\omega^2_j\omega^2_k K'^2_{jk}\right)$$

Mean square energy fluctuation in range $(\omega, \omega + \Delta\omega)$ in segment $(0, a)$:

$$\overline{\Delta E^2_{(\omega, a)}} = \frac{1}{8}\sum_{j \neq k}(\overline{\dot{q}_j^2}\,\overline{\dot{q}_k^2}K_{jk}^2 + \overline{q_j^2}\,\overline{q_k^2}\omega_j^2\omega_k^2K'^2_{jk})$$

Virial theorem for oscillator, $\overline{E_{\text{kin}}} = \overline{E_{\text{pot}}} = \frac{1}{2}E_{\text{tot}}$: $\frac{l}{4}\overline{\dot{q}_j^2} = \frac{l}{4}\omega_j^2\overline{q_j^2} = \frac{1}{2}H_j\left(m = \frac{l}{2}\right)$.

Hence, $\overline{\dot{q}_j^2} = \frac{2}{l}H_j$ and $\overline{q_j^2} = \frac{2}{l\omega_j^2}H_j$:

$$\boldsymbol{\overline{\Delta E^2_{(\omega, a)}} = \frac{1}{2l^2}\sum_{j \neq k}H_jH_k(K_{jk}^2 + K'^2_{jk})}$$

From discrete sums to integrals (requires extra assumption: **distribution of energy over frequency is smooth, i.e., $H_j$ varies smoothly with $j$**): $\sum_j \to \frac{l}{\pi}\int d\omega \left(\text{since } \omega_j = j\frac{\pi}{l}\right)$, $\sum_k \to \frac{l}{\pi}\int d\omega'$, $H_jH_k \to H_\omega H_{\omega'}$, $K_{jk}^2 \to K^2_{\omega\omega'} \to \pi a\delta(\omega - \omega')$:*

$$\overline{\Delta E^2_{(\omega, a)}} = \frac{1}{2l^2}\int d\omega\int d\omega'\left(\frac{l}{\pi}\right)^2 2\pi a\delta(\omega - \omega')H_\omega H_{\omega'} = \boldsymbol{\frac{a}{\pi}\int d\omega H_\omega^2}$$

* $\int d\omega' f(\omega')K^2_{\omega\omega'} = \int d\omega' f(\omega')\frac{\sin^2((\omega - \omega')a)}{(\omega - \omega')^2} = \int d\omega' f(\omega')\pi a\delta(\omega - \omega') = \pi a f(\omega)$

Mean square energy fluctuation in segment $(0, a)$ …

… in range $(\omega, \omega + \Delta\omega)$:

$$\overline{\Delta E^2_{(\omega, a)}} = \frac{a}{\pi}\int d\omega H^2_\omega$$

$$\longrightarrow$$

$$f(\omega) \rightarrow f(\nu)$$

$$\int d\omega \rightarrow \times 2\pi\Delta\nu$$

… in range $(\nu, \nu + \Delta\nu)$:

$$\boxed{\overline{\Delta E^2_{(\nu, a)}} = 2a\Delta\nu H^2_\nu = \frac{\overline{E_{(\nu, a)}}^2}{2a\Delta\nu}}$$

$$\overline{E_{(\nu, a)}} = \frac{a}{l}H_\nu N_\nu$$

$N_\nu$ = number of modes in $(\nu, \nu + \Delta\nu)$

$$N_\nu \frac{\pi}{l} = \Delta\omega = 2\pi\Delta\nu \rightarrow N_\nu = 2l\Delta\nu$$

$$\overline{E_{(a, \nu)}} = 2a\Delta\nu H_\nu$$

Mean square energy fluctuations proportional to mean energy squared.

Result for classical waves.

Comment: This is a very general result (already found by Lorentz in 1912/16). It holds for any smooth distribution of energy over frequency (equilibrium or not). Temperature does not enter into the derivation at all.

## 2. Quantum calculation of the mean square fluctuation of the energy in a narrow frequency range $(\omega, \omega + \Delta\omega)$ in a small segment $(0, a)$ of the string.

**Preview**: All quantities—in modern terms—become operators. Because operators $q$ and $\dot{q}$ don’t commute, the quantum analogue of the cross terms $\overline{\Delta E_1 \Delta E_2} + \overline{\Delta E_2 \Delta E_1}$ contributes to the mean square energy fluctuation. This contribution is essential for correctly reproducing the particle term in the Einstein fluctuation formula.

**Key point of Heisenberg’s *Umdeutung* paper**: operators representing position and momentum still satisfy the old equations of motion → Operator $q_k(t)$ for quantum harmonic oscillator satisfies classical equation $\ddot{q}_k(t) = -\omega_k^2 q_k(t)$. Solution:

$$q_k(t) = q_k(0)\cos(\omega_k t) + \frac{2p_k(0)}{l\omega_k}\sin(\omega_k t)$$

$$\dot{q}_k(t) = \frac{2p_k(0)}{l}\cos(\omega_k t) - \omega_k q_k(0)\sin(\omega_k t)$$

Operator for mean square energy fluctuation in range $(\omega, \omega + \Delta\omega)$ in segment $(0, a)$:

$$\overline{\Delta H^2_{(\omega, a)}} = \overline{\Delta H^2_{1(\omega, a)}} + \overline{\Delta H^2_{2(\omega, a)}} + \overline{\Delta H_{1(\omega, a)} \Delta H_{2(\omega, a)}} + \overline{\Delta H_{2(\omega, a)} \Delta H_{1(\omega, a)}}$$

Consider cross terms (which vanished classically):

$$\overline{\Delta H_{1(\omega, a)} \Delta H_{2(\omega, a)}} + \overline{\Delta H_{2(\omega, a)} \Delta H_{1(\omega, a)}}$$

$$= \frac{1}{16} \sum_{j \neq k} \sum_{j' \neq k'} (\overline{\dot{q}_j \dot{q}_k q_{j'} q_{k'}} \omega_{j'} \omega_{k'} K_{jk} K'_{j'k'} + \overline{q_j q_k \dot{q}_{j'} \dot{q}_{k'}} \omega_j \omega_k K'_{jk} K_{j'k'})$$

$$= \frac{1}{8} \sum_{j \neq k} (\overline{\dot{q}_j q_j}\,\overline{\dot{q}_k q_k} + \overline{q_j \dot{q}_j}\,\overline{q_k \dot{q}_k}) \omega_j \omega_k K_{jk} K'_{jk}$$

**No longer vanishes since $\dot{q}_j$ and $q_j$ do not commute!**

Evaluate $\overline{\dot{q}_k q_k}$:

$$\overline{\dot{q}_k q_k} = \overline{\left(\frac{2p_k(0)}{l} \cos(\omega_k t) - \omega_k q_k(0) \sin(\omega_k t)\right)\left(q_k(0) \cos(\omega_k t) + \frac{2p_k(0)}{l\omega_k} \sin(\omega_k t)\right)}$$

$$= \frac{2}{l}(p_k(0) q_k(0) - q_k(0) p_k(0)) \frac{1}{2} = -\frac{i\hbar}{l}$$

Hence:

$$\overline{\Delta H_{1(\omega, a)} \Delta H_{2(\omega, a)}} + \overline{\Delta H_{2(\omega, a)} \Delta H_{1(\omega, a)}} = -\frac{1}{l^2} \sum_{j \neq k} \frac{\hbar^2}{4} \omega_j \omega_k K_{jk} K'_{jk}$$

Operator for mean square energy fluctuations in range $(\omega, \omega + \Delta\omega)$ in segment $(0, a)$:

**Contribution from cross terms** (because of non-commutativity of $p$ and $q$):

$$\overline{\Delta H_{1_{(\omega, a)}} \Delta H_{2_{(\omega, a)}}} + \overline{\Delta H_{2_{(\omega, a)}} \Delta H_{1_{(\omega, a)}}} = -\frac{1}{l^2} \sum_{j \neq k} \frac{h^2}{4} \omega_j \omega_k K_{jk}^2$$

**Contribution from square terms** (same as before)

$$\overline{\Delta H^2_{1_{(\omega, a)}}} + \overline{\Delta H^2_{2_{(\omega, a)}}} = \frac{1}{l^2} \sum_{j \neq k} H_j H_k K_{jk}^2$$

Note: $K_{jk}^2$, $K'^2_{jk}$, and $K_{jk} K'_{jk}$ can be used interchangeably in these sums

**Added together**

$$\overline{\boldsymbol{\Delta H^2_{(\omega, a)}}} = \frac{\mathbf{1}}{\boldsymbol{l^2}} \sum_{j \neq k} \left( \boldsymbol{H_j H_k} - \frac{\boldsymbol{\hbar^2}}{\mathbf{4}} \boldsymbol{\omega_j \omega_k} \right) \boldsymbol{K_{jk}^2}$$

From discrete sums to integrals (on the assumption that $H_j$ varies smoothly with $j$):

$$\sum_{j \neq k} \rightarrow \left(\frac{l}{\pi}\right)^2 \int d\omega \int d\omega' ,\ H_j H_k \rightarrow H_\omega H_{\omega'},\ K_{jk}^2 \rightarrow K^2_{(\omega\omega')} \rightarrow \pi a \delta(\omega - \omega'):$$

$$\overline{\boldsymbol{\Delta H^2_{(\omega, a)}}} = \frac{1}{l^2} \int d\omega \int d\omega' \left(\frac{l}{\pi}\right)^2 \left( H_\omega H_{\omega'} - \frac{\hbar^2}{16\pi^2} \omega\omega' \right) \pi a \delta(\omega - \omega') = \frac{\boldsymbol{a}}{\boldsymbol{\pi}} \int \boldsymbol{d\omega} \left( \boldsymbol{H_\omega^2} - \left( \frac{\boldsymbol{\hbar\omega}}{\mathbf{2}} \right)^2 \right)$$

**Operator** for mean square energy fluctuations in segment $(0, a)$ …

… in range $(\omega, \omega + \Delta\omega)$: … in range $(\nu, \nu + \Delta\nu)$:

$$\overline{\Delta H^2_{(\omega, a)}} = \frac{a}{\pi}\int d\omega\left(H^2_\omega - \left(\frac{\hbar\omega}{2}\right)^2\right) \xrightarrow[f(\omega) \to f(\nu)]{} \boldsymbol{\overline{\Delta H^2_{(\nu, a)}} = 2a\Delta\nu\left(H^2_\nu - \left(\frac{h\nu}{2}\right)^2\right)}$$

$$\int d\omega \to \ \times 2\pi\Delta\nu$$

**Expectation value** of mean square energy fluctuation in $(\nu, \nu + \Delta\nu)$ in $(0, a)$

$$\boxed{\boldsymbol{\overline{\Delta E^2_{(\nu, a)}}}} \equiv \langle\{n_\nu\}|\overline{\Delta H^2_{(\nu, a)}}|\{n_\nu\}\rangle \overset{(1)}{=} 2a\Delta\nu\left(\left(n_\nu + \frac{1}{2}\right)^2 - \frac{1}{4}\right)h^2\nu^2$$

$$\overset{(2)}{=} 2a\Delta\nu((n_\nu h\nu)^2 + (n_\nu h\nu)h\nu)$$

$$\overset{(3)}{=} \boxed{\boldsymbol{\frac{\overline{E_{(\nu, a)}}^2}{2a\Delta\nu} + \overline{E_{(\nu, a)}}h\nu}}$$

Exactly the same form as the Einstein fluctuation formula with wave and particle term!

(1) $\langle\{n_\nu\}|H_\nu|\{n_\nu\}\rangle = \left(n_\nu + \frac{1}{2}\right)h\nu$ with $\frac{h\nu}{2}$ the zero-point energy

(2) $2^{\text{nd}}$ term, coming from $[p, q] \neq 0$, cancels the square of the zero-point energy coming from $1^{\text{st}}$ term.

(3) ‘Renormalized’ energy (with zero-point energy subtracted): $\overline{E_{(\nu, a)}} \equiv \frac{a}{l}(n_\nu h\nu)N_\nu = 2a\Delta\nu(n_\nu h\nu)$

Number of modes in in $(\nu, \nu + \Delta\nu)$: $N_\nu = 2l\Delta\nu$

Expectation value of the mean square fluctuation of the energy in $(\nu, \nu + \Delta\nu)$ in $(0, a)$

$$\overline{\Delta E^2_{(\nu, a)}} = \frac{\overline{E_{(\nu, a)}}^2}{2a\Delta\nu} + \overline{E_{(\nu, a)}}h\nu$$

Comments:

- Since the entire derivation is for a finite frequency range, there are no problems with infinities (*pace*, e.g., Jürgen Ehlers at HQ0, Mehra-Rechenberg, Vol. 3, p. 156).
- Like its classical counterpart, this quantum formula is a very general result. It holds for all states $\{n_\nu\}$ in which $n_\nu$ is a reasonably smooth function of $\nu$. There is no restriction to equilibrium states. Again, temperature does not enter into the derivation.
- The formula is for the **quantum spread** of the energy in a narrow frequency range in a small segment of the string in an energy eigenstate of the whole system. The operators $H$ and $H_{(\omega, a)}$ do not commute. The system is in an eigenstate of the full Hamiltonian $H$ but in a superposition of eigenstates of $H_{(\omega, a)}$.
- To get from the formula for the **quantum spread** to a formula for **thermodynamical fluctuations**, we need to go from energy eigenstates to thermal averages of such states. It can be shown that this transition does not affect the form of the formula (key point: formula only involves products of thermally independent modes).

Claim: The formula for the thermodynamical fluctuation of the energy in a narrow frequency range in a small subvolume has the exact same form as the formula for the quantum spread in that energy.

Thermal average for operator $O$: $$[\langle\{n_i\}|O|\{n_i\}\rangle] = \frac{\sum_{\{n_i\}} \langle\{n_i\}|O|\{n_i\}\rangle e^{-\beta E_{\{n_i\}}}}{\sum_{\{n_i\}} e^{-\beta E_{\{n_i\}}}}$$

- Square brackets for canonical ensemble expectation value
- $\beta = \frac{1}{kT}$ Boltzmann factor
- $E_{\{n_i\}} = \sum_i \left(n_i + \frac{1}{2}\right) h\nu_i$

**Naively:** $\langle\{n_\nu\}|\overline{\Delta H^2_{(\nu,\, a)}}|\{n_\nu\}\rangle$ $\left[\langle\{n_\nu\}|\overline{\Delta H^2_{(\nu,\, a)}}|\{n_\nu\}\rangle\right]$

quantum spread → thermal average

$$\overline{\Delta E^2_{(\nu,\, a)}} = \frac{\overline{E_{(\nu,\, a)}}^2}{2a\Delta\nu} + \overline{E_{(\nu,\, a)}}h\nu \qquad \left[\overline{\Delta E^2_{(\nu,\, a)}}\right] = \frac{[\overline{E_{(\nu,\, a)}}^2]}{2a\Delta\nu} + [\overline{E_{(\nu,\, a)}}]h\nu$$

**Wrong result:** $[\overline{E_{(\nu,\, a)}}^2]$ should be $[\overline{E_{(\nu,\, a)}}]^2$!

Calculate $\left[\langle\{n_i\}|\overline{\Delta H^2_{(\nu,\,a)}}|\{n_i\}\rangle\right]$ with $\overline{\Delta H^2_{(\nu,\,a)}} = \frac{1}{l^2}\sum_{j\neq k}\left(H_jH_k - \frac{h^2}{4}\nu_j\nu_k\right)K^2_{jk}$

Only non-trivial part:

$$[\langle\{n_i\}|H_jH_k|\{n_i\}\rangle] = \frac{\sum_{n_j}\sum_{n_k}\left(n_j+\frac{1}{2}\right)h\nu_j\left(n_k+\frac{1}{2}\right)h\nu_k e^{-\beta\left(\left(n_j+\frac{1}{2}\right)h\nu_j+\left(n_k+\frac{1}{2}\right)h\nu_k\right)}}{\sum_{n_j}\sum_{n_k} e^{-\beta\left(\left(n_j+\frac{1}{2}\right)h\nu_j+\left(n_k+\frac{1}{2}\right)h\nu_k\right)}}$$

Product of factor for $j^{\text{th}}$ mode and factor for thermally independent $k^{\text{th}}$ mode:

$$\frac{\sum\left(n_j+\frac{1}{2}\right)h\nu_j e^{-\beta\left(n_j+\frac{1}{2}\right)h\nu_j}}{\sum e^{-\beta\left(n_j+\frac{1}{2}\right)h\nu_j}} = \left(\hat{n}_j+\frac{1}{2}\right)h\nu_j \text{ with: } \hat{n}_j \equiv \frac{1}{e^{\beta h\nu_j}-1}$$

(Planck function for average thermal excitation level of $j^{\text{th}}$ mode)

Insert:

$$\left[\langle\{n_i\}|\overline{\Delta H^2_{(\nu,\,a)}}|\{n_i\}\rangle\right] = \frac{1}{l^2}\sum_{j\neq k}\left(\hat{n}_j\hat{n}_k+\frac{1}{2}(\hat{n}_k+\hat{n}_k)\right)h^2\nu_j\nu_kK^2_{jk}$$

Claim: The formula for the **thermodynamical fluctuation** of the energy in a narrow frequency range in a small subvolume has the exact same form as the formula for the **quantum spread** in that energy.

Formula for discrete sums:

$$\left[\langle\{n_i\}|\overline{\Delta H^2_{(\nu,\,a)}}|\{n_i\}\rangle\right] = \frac{1}{l^2}\sum_{j\neq k}\left(\hat{n}_j\hat{n}_k+\frac{1}{2}(\hat{n}_k+\hat{n}_k)\right)h^2\nu_j\nu_k K^2_{jk}$$

From discrete sums to integrals (no extra assumption needed: $\hat{n}_j$ varies smoothly with $\nu_j$)

$$[\overline{\Delta E^2_{(\nu,\,a)}}] \equiv \left[\langle\{n_\nu\}|\overline{\Delta H^2_{(\nu,\,a)}}|\{n_\nu\}\rangle\right] = 2a\Delta\nu(\hat{n}^2_\nu+\hat{n}_\nu)h^2\nu^2$$

Use $[\overline{E_{(\nu,\,a)}}] = 2a\Delta\nu(\hat{n}_\nu h\nu)$:

$$[\overline{\Delta E^2_{(\nu,\,a)}}] = \frac{[\overline{E_{(\nu,\,a)}}]^2}{2a\Delta\nu} + [\overline{E_{(\nu,\,a)}}]h\nu \qquad \text{thermodynamical fluctuation}$$

$$\overline{\Delta E^2_{(\nu,\,a)}} = \frac{\overline{E_{(\nu,\,a)}}^2}{2a\Delta\nu} + \overline{E_{(\nu,\,a)}}h\nu \qquad \text{quantum spread}$$

Two formulae have exactly the same structure.

# Jordan on the importance of his result

Conclusion of 3M: "If one bears in mind that the question considered here is actually somewhat remote from the problems whose investigation led to the growth of quantum mechanics, the result … can be regarded as particularly encouraging for the further development of the theory."

"The basic difference between the theory proposed here and that used hitherto … lies in the characteristic kinematics and not in a disparity of the mechanical laws. One could indeed perceive one of the most evident examples of the difference between quantum-theoretical kinematics and that existing hitherto on examining [the quantum fluctuation formula], which actually involves no mechanical principles whatsoever [sic]."

## Jordan on the importance of his result

From Jordan’s 1928 history of the light-quantum hypothesis: “it turned out to be superfluous to explicitly adopt the light-quantum hypothesis: We explicitly stuck to the wave theory of light and only changed the kinematics of cavity waves quantum-mechanically. From this, however, the characteristic light-quantum effects emerged automatically as a consequence … This is a whole new turn in the light-quantum problem. It is not necessary to include the picture [*Vorstellung*] of light quanta among the assumptions of the theory. One can—and this seems to be the natural way to proceed—start from the wave picture. If one formulates this with the concepts of quantum mechanics, then the characteristic light-quantum effects emerge as necessary consequences from the general laws [*Gesetzmäßigkeiten*] of quantum theory.”

From the 1930 Born-Jordan book on matrix mechanics: “Consideration of the Wien limit of the Planck radiation law suggests, according to Einstein, that the wave picture has to be replaced or supplemented by the particle picture. Quantum mechanics, however, makes it possible to restore agreement with Planck's law and all its consequences *without* giving up the wave picture. It suffices to work through the model [*Modellvorstellung*] of the classical wave theory with the exact quantum-theoretical kinematics and mechanics. The characteristic light-quantum effects then emerge automatically, without the addition of new hypotheses, as necessary consequences of the wave theory.”

# Conclusions

## Jordan’s derivation of Einstein’s fluctuation formula in 3M

### The derivation

- is tricky but kosher.

### The result

- resolves the conundrum of the wave-particle duality for light;
- was not hailed as a major breakthrough in its own right in 3M (and Jordan’s subsequent expositions did not get much attention);
- can be seen as the birth of quantum field theory.