Projective Geometry and the Origins of the Dirac Equation

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‘Dirac’s Hidden Geometry’ and the Dirac Equation

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“the solution came rather, I would say, out of the blue” (Dirac, 1977)
Outline

- Projective Geometry and $q$-numbers
  - Connection is weak
  - Inconsistent with Dirac's comments
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  - Encouraged by Dirac's comments
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- The Dirac Equation
  - A role for projective geometry?
  - Archival evidence
An unusual route for a physicist:
Dirac’s Education

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Dirac had a background in pure mathematics.
Dirac's Introduction to Projective Geometry

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Dirac was keen to speak about his fondness for projective geometry. Is there a connection to his work in quantum mechanics?
q-numbers and c-numbers

Timeline for Dirac’s Quantum Mechanics

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- 1926a ‘Quantum Mechanics and a Preliminary Investigation of the Hydrogen Atom’
- 1926b ‘On Quantum Algebra’
- 1927 ‘The Physical Interpretation of Quantum Mechanics’
Definition of q-numbers

At present one can form no picture of what a q-number is like. One cannot say that one q-number is greater or less than another. All one knows about q-numbers is that if \( z_1 \) and \( z_2 \) are two q-numbers, or one q-number and one c-number, there exist the numbers \( z_1 + z_2, z_1z_2, z_2z_1 \), which will in general be q-numbers but may be c-numbers. One knows nothing of the processes by which the numbers are formed except that they satisfy all the ordinary laws of algebra, excluding the commutative law of multiplication, i.e.,

\[
\begin{align*}
z_1 + z_2 &= z_2 + z_1, \\
(z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3), \\
(z_1z_2)z_3 &= z_1(z_2z_3), \\
z_1(z_2 + z_3) &= z_1z_2 + z_1z_3, \quad (z_1 + z_2)z_3 &= z_1z_3 + z_2z_3,
\end{align*}
\]

and if

\[
z_1z_2 = 0,
\]

either

\[
z_1 = 0 \quad \text{or} \quad z_2 = 0;
\]

but

\[
z_1z_2 \neq z_2z_1,
\]

in general, except when \( z_1 \) or \( z_2 \) is a c-number.
A Role for Projective Geometry?

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3. Dirac used projective geometry as a means to visualize $q$-numbers. (Mehra and Rechenberg, Kragh 1981)
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4. Dirac’s quantum algebra was essentially geometrical (Rechenberg 1987)
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Dirac’s primary role for projective geometry was as a means of visualization for Minkowski space and Lorentz transformations NOT \( q \)-numbers.
AHQP interview with Kuhn (1962)

"All my work since then [Bristol] has been very much of a geometrical nature, rather than of an algebraic nature."

Puzzled, Kuhn asks later (1963) if Dirac regards his "peculiar q-number manipulations ... as being algebraic rather than geometric."

"Yes, but I only used them in an elementary way."

So what is the connection to projective geometry?
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Projective Geometry and Special Relativity

“Four dimensions were very popular then for the geometrists to work with. It was all done with the notions of projective geometry rather than metrical geometry. So I became very familiar with that kind of mathematics in that way. I’ve found it useful since then in understanding the relations which you can have in Minkowski space. You can picture all the directions in Minkowski space as the points in a three-dimensional vector space. I always used these geometrical ideas for getting clear notions about relationships in relativity although I didn’t refer to them in my published works.” (ibid.)
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Kuhn asks, anything to do with quantum mechanics?

“No. It doesn’t connect at all with non-commutative
A New Role for Projective Geometry

Dirac repeatedly emphasized connection to Minkowski space (Trieste & Boston 1972).
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1. Dirac was trying to find a Lorentz invariant wave equation.

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Did Dirac use projective geometry in his search for the Dirac equation?
What is Projective Geometry?

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- Any two lines meet at a unique point.
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The study of geometrical properties invariant under projection.

- Any two lines meet at a unique point.
- Parallel lines meet at a point at infinity.
An Illustration
Homogeneous Co-ordinates

- A projective space contains the points at infinity.
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- A point in real projective space $RP^n$ has $n + 1$ homogeneous co-ordinates.
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- Point in real projective plane has 3 co-ords $(x_1, x_2, x_3)$.
- Consider a point $(y_1, y_2)$ with $y_1 = \frac{x_1}{x_3}$, $y_2 = \frac{x_2}{x_3}$ so that $(x_1, x_2, x_3) \equiv (cx_1, cx_2, cx_3)$. 
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- The points $(x_1, x_2, 0)$ form the line at infinity, approached from either direction.
Conic Sections

Conic sections 1) Parabola, 2) Circles and Ellipses, 3) Hyperbolae.
Conics

- Under projection, circle not mapped to circle but conic mapped to conic.
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- Quadrics generalise the conic to higher dimensional spaces.
Minkowski’s Address

Minkowski (1908): tip of velocity vector constrained to surface of a hyperboloid $t^2 - x^2 - y^2 - z^2 = 1$. 
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Lorentz transformation leaves hyperboloid invariant.
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- Correspond to: points inside, outside, or on the absolute quadric.
- Lorentz transformations leave the absolute quadric invariant.
The Absolute Quadric

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In the case of Minkowski space, defined by the Minkowski metric:

\[
\eta_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0
\]
“if we just think in terms of this hyperplane at infinity, we have a three-dimensional space. Talking of a four dimensional space is something that is hard to imagine, but we can’t really imagine it. We talk about it as though we could, but when we are concerned just with directions, the things in the space of physics, we can represent them all in terms of a three-dimensional space according to the methods of projective geometry. We have a three-dimensional projective space in which there is an absolute quadric.” (Dirac, 1972)
Dirac’s Task

Dissatisfied with Klein-Gordon equation

\[
\left[ (i\hbar \frac{\partial}{\partial t} + \frac{e}{c} A_0)^2 + \sum_r \left( -i\hbar \frac{\partial}{\partial x^r} + \frac{e}{c} A_r \right)^2 + m^2 c^2 \right] \psi = 0.
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What Dirac wants:

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What Dirac wants:

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2. First order in time, so first order in the momenta.
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What Dirac wants:

1. Wave equation invariant under Lorentz transformation.
2. First order in time, so first order in the momenta.
3. Agrees with Klein-Gordon equation.
Dirac’s Story

“Playing around with mathematics” he noticed the “very pretty mathematical result” (Dirac, 1972)

\[(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)^2 = p_1^2 + p_2^2 + p_3^2.\]
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“It took me quite a while, studying over this dilemma ...”

Came to realize that 4 × 4 matrices would suffice.
The symmetry between \( p_0 \) and \( p_1, p_2, p_3 \) required by relativity shows that, since the Hamiltonian we want is linear in \( p_0 \), it must also be linear in \( p_1, p_2 \) and \( p_3 \). Our wave equation is therefore of the form

\[
(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta) \psi = 0, \tag{4}
\]

Equation (4) leads to

\[
0 = (-p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta) (p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta) \psi = \left[ -p_0^2 + \sum \alpha_i^2 p_i^2 + \sum (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1) p_1 p_2 + \beta^2 + \sum (\alpha_1 \beta + \beta \alpha_1) p_1 \right] \psi, \tag{5}
\]

where the \( \Sigma \) refers to cyclic permutation of the suffixes 1, 2, 3. This agrees with (3) if

\[
\begin{align*}
\alpha_r^2 &= 1, & \alpha_r \alpha_s + \alpha_s \alpha_r &= 0 \quad (r \neq s) \\
\beta^2 &= m^2c^2, & \alpha_r \beta + \beta \alpha_r &= 0
\end{align*}
\]

If we put \( \beta = \alpha_4 mc \), these conditions become

\[
\alpha_\mu^2 = 1, \quad \alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0 \quad (\mu \neq \nu) \quad (\mu, \nu = 1, 2, 3, 4). \tag{6}
\]

We can suppose the \( \alpha_{\mu} \)'s to be expressed as matrices in some matrix scheme,
We must now find four matrices \( \sigma \) to satisfy the conditions (6). We make use of the matrices

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

which Pauli introduced\(^*\) to describe the three components of spin angular momentum. These matrices have just the properties

\[
\sigma_r^2 = 1, \quad \sigma_r \sigma_s + \sigma_s \sigma_r = 0, \quad (r \neq s),
\]

that we require for our \( \omega \)'s. We cannot, however, just take the \( \sigma \)'s to be three of our \( \omega \)'s, because then it would not be possible to find the fourth. We must extend the \( \sigma \)'s in a diagonal manner to bring in two more rows and columns, so that we can introduce three more matrices \( \rho_1, \rho_2, \rho_3 \) of the same form as \( \sigma_1, \sigma_2, \sigma_3 \), but referring to different rows and columns, thus:

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
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\rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
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If we now take

\[
\begin{align*}
\alpha_1 &= \rho_1 \sigma_1, & \alpha_2 &= \rho_1 \sigma_2, & \alpha_3 &= \rho_1 \sigma_3, & \alpha_4 &= \rho_3,
\end{align*}
\]

all the conditions (6) are satisfied, e.g.,

\[
\begin{align*}
\alpha_1^2 &= \rho_1 \sigma_1 \rho_1 \sigma_1 = \rho_1^2 \sigma_1^2 = 1, \\
\alpha_2 \alpha_2 &= \rho_1 \sigma_1 \rho_1 \sigma_2 = \rho_1^2 \sigma_1 \sigma_2 = -\rho_1^2 \sigma_2 \sigma_1 = -\alpha_2 \alpha_1.
\end{align*}
\]
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Dirac’s Method

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What We Know

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In particular, did he use projective geometry?
\[ F\psi \equiv \left[ (\alpha_{5\mu} + i\alpha_{6\mu}) (d_{\mu} + iA_{\mu}) + mc \right] \psi = 0 \]  \hspace{1cm} (1)

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- Dirac has found here the general form of the equation he seeks.
- No explicit (anti)-commutation relations.
- Nature of the \( \alpha \)'s unclear, but not \( c \)-numbers.
Agrees with Klein-Gordon equation.
Comments

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- Dirac knows the commutation properties of his $\alpha$'s.
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- Dirac knows the commutation properties of his $\alpha$’s.
- Explicit $4 \times 4$ matrix representation.
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Where did they come from?
Best guess:
Comments

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- Linear equation of a line between $y$ and $z$, defines a linear complex.
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- Sets up system of 4 equations defining a line.
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- Expressed in terms of $\alpha$’s.
- Sets up system of 4 equations defining a line.
- In Klein (1870) co-ordinates, this has a general quadratic form.
Did Projective Geometry Lead to the Dirac Equation?

Dirac begins by considering projective geometry (p. 2). No matrix representation of $\alpha$'s until p. 7. Pauli matrices appear after Dirac matrices. Was he guided by geometrical, not algebraic reasoning?
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Begins considering properties of $2 \times 2$ matrices.
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\frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{\partial}{\partial x} \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} + \frac{\partial}{\partial z} \begin{pmatrix} i\psi_2 \\ -i\psi_1 \end{pmatrix} \\
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Tries to find $3 \times 3$ matrices with these properties.
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Tries to find $3 \times 3$ matrices with these properties.

Finds suitable $4 \times 4$ matrices - leads to $\alpha$'s on p. 7.
Dirac and Projective Geometry

Dirac did not use projective geometry in his early work on QM. Projective geometry was primarily a means for visualisation of Minkowski space. Definite mathematical correspondence and clear role.

Dirac used projective geometry in his search for the Dirac equation.

Tom Pashby
Projective Geometry and the Origins of the Dirac Equation
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The Dirac Equation

The Dirac Equation

Relevant manuscript source exists. Much of it unclear, including order of pages. The “playing around with mathematics” involved projective geometry, but no need for major revision. Dirac was right: he did not consider the Pauli equation and spin, although he did try two component wave functions. The realization that $\alpha$'s were analogous to Pauli matrices led straight to the solution - no delay. Dirac did consider $3 \times 3$ matrices (contra Mehra and Rechenberg, 2000).
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